# FOURIER SERIES

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***Abstract*— A In mathematics, a Fourier series is a way to represent a wave-like function as a combination of simple sine waves. More formally, it decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sine and cosine (or, equivalently, complex exponentials). The Discrete-time Fourier transform is a periodic function, often defined in terms of a Fourier series. The Z-transform, another example of application, reduces to a Fourier series for the important case |z|=1. Fourier series are also central to the original proof of the Nyquist–Shannon sampling theorem [1]. The study of Fourier series is a branch of Fourier analysis. an infinite trigonometric series of the form 1/2a0 + a1cos x + b1sin x + a2cos 2x + b2sin 2x + …, where a0, a1, b1, a2, b2 … are the Fourier coefficients. It is used, especially in mathematics and physics, to represent or approximate any periodic function by assigning suitable values to the coefficients. there are some of the conditions of Fourier series and in that there is an Euler's formula also .**

***Index Terms*- Fourier series, conditions, Euler's Formulae, expansion.**

## I. INTRODUCTION

The heat equation is a partial differential equation. Prior to Fourier's work, no solution to the heat equation was known in the Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolution. This superposition or linear combination is called the Fourier series.From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century. Later, Peter Gustav Lejeune Dirichle and Bernhard Riemann expressed Fourier's results with greater precision and formality.Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigensolutions are sinusoids. The Fourier series has many such applications in electrical engineering, vibration analysis, acoustic, optics, signal processing, image processing, quantum mechanics, econometrics thin-walled shell theory, etc.general case, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolution. This superposition or linear combination is called the Fourier series.From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century. Later, Peter Gustav Lejeune Dirichle and Bernhard Riemann expressed Fourier's results with greater precision and formality.Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigensolutions are sinusoids. The Fourier series has many such applications in electrical engineering, vibration analysis, acoustic, optics, signal processing, image processing, quantum mechanics, econometric thin-walled shell theory, etc.

In this section, $s(x)$ denotes a function of the real variable $x$, and $s$ is integrable on an interval $[x_0, x_0 + P]$, for real numbers $x_0$ and $P$. We will attempt to represent $s$ in that interval as an infinite sum, or series, of harmonically related sinusoidal functions. Outside the interval, the series is periodic with period $P$ (frequency $1/P$). It follows that if $s$ also has that property, the approximation is valid on the entire real line. We can begin with a finite summation (or *partial sum*)**:**

$$s_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cdot \sin\left(\tfrac{2\pi nx}{P} + \phi_n\right), \quad \text{for integer } N \geq 1.$$

$S_N(x)$ is a periodic function with period **P**. Using the identities**:**

$$\sin\left(\tfrac{2\pi nx}{P}+\phi_n\right) \equiv \operatorname{Re}\left\{\frac{1}{i}\cdot e^{i\left(\frac{2\pi nx}{P}+\phi_n\right)}\right\} = \frac{1}{2i}\cdot e^{i\left(\frac{2\pi nx}{P}+\phi_n\right)} + \left(\frac{1}{2i}\cdot e^{i\left(\frac{2\pi nx}{P}+\phi_n\right)}\right)^*,$$

$$\begin{aligned} s_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N}\left(\overbrace{a_n}^{A_n\sin(\phi_n)}\cos\left(\tfrac{2\pi nx}{P}\right) + \overbrace{b_n}^{A_n\cos(\phi_n)}\sin\left(\tfrac{2\pi nx}{P}\right)\right) \\ &= \sum_{n=-N}^{N} c_n\cdot e^{i\frac{2\pi nx}{P}}, \end{aligned}$$

we can also write the function in these equivalent forms**:**

$$c_n \stackrel{\text{def}}{=} \begin{cases} \frac{A_n}{2i}e^{i\phi_n} = \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ \frac{1}{2}a_0 & \text{for } n = 0 \\ c^*_{|n|} & \text{for } n < 0. \end{cases}$$

**Where:** When the coefficients (known as **Fourier coefficients**) are computed as follows**:**

$$a_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x)\cdot\cos\left(\tfrac{2\pi nx}{P}\right)\,dx$$

$s_N(x)$ approximates $s(x)$ on $[x_0,\ x_0+P]$, and the approximation improves as $N \to \infty$. The infinite sum, $s_\infty(x)$, is called the **Fourier series** representation of $s$. In engineering applications, the Fourier series is generally presumed to converge everywhere except at discontinuities, since the functions encountered in engineering are more well behaved than the ones that mathematicians can provide as counter-examples to this presumption. In particular, the Fourier series converges absolutely and uniformly to $s(x)$ whenever the derivative of $s(x)$ (which may not exist everywhere) is square integrable. If a function is square-integrable on the interval $[x_0, x_0+P]$, then the Fourier series converges to the function at *almost every* point. See Convergence of Fourier series. It is possible to define Fourier coefficients for more general functions or distributions, in such cases convergence in norm or weak convergence is usually of interest.

## II. FOURIER SERIES

### I. A simple Fourier series

We now use the formula above to give a Fourier series expansion of a very simple function. Consider a sawtooth wave:

$$s(x) = \frac{x}{\pi}, \quad \text{for} -\pi < x < \pi,$$

In this case, the Fourier coefficients are given by

It can be proven that the Fourier series converges to $s(x)$ at every point $x$ where $s$ is differentiable, and therefore:

$$\begin{aligned} s(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx), \quad \text{for} \quad x - \pi \notin 2\pi\mathbf{Z}. \end{aligned}$$

When $x = \pi$, the Fourier series converges to 0, which is the half-sum of the left- and right-limit of $s$ at $x = \pi$. This is a particular instance of the Dirichlet theorem for Fourier series.

## II. FOURIER'S MOTIVATION

The Fourier series expansion of our function in example 1 looks much less simple than the formula $s(x) = x/\pi$, and so it is not immediately apparent why one would need this Fourier series. While there are many applications, we cite Fourier's motivation of solving the heat equation. For example, consider a metal plate in the shape of a square whose side measures $\pi$ meters, with coordinates $(x, y) \in [0, \pi] \times [0, \pi]$. If there is no heat source within the plate, and if three of the four sides are held at 0 degrees Celsius, while the fourth side, given by $y = \pi$, is maintained at the temperature gradient $T(x, \pi) = x$ degrees Celsius, for $x$ in $(0, \pi)$, then one can show that the stationary heat distribution (or the heat distribution after a long period of time has elapsed) is given by

$$T(x,y) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)\frac{\sinh(ny)}{\sinh(n\pi)}.$$

Here, sinh is the hyperbolic sin function. This solution of the heat equation is obtained by multiplying each term of **Eq.1** by $\sinh(ny)/\sinh(n\pi)$. While our example function $s(x)$ seems to have a needlessly complicated Fourier series, the heat distribution $T(x, y)$ is nontrivial. The function $T$ cannot be written as a closed-form expression. This method of solving the heat problem was made possible by Fourier's work.

## III. EXTENSIONS

Fourier series on a square :

We can also define the Fourier series for functions of two variables $x$ and $y$ in the square $[-\pi, \pi]\times[-\pi, \pi]$:

$$c_{j,k} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-ijx}e^{-iky}\,dx\,dy.$$

Aside from being useful

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cos(nx)\,dx = 0, \quad n \ge 0.$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\sin(nx)\,dx \\ &= -\frac{2}{\pi n}\cos(n\pi) + \frac{2}{\pi^2 n^2}\sin(n\pi) \\ &= \frac{2\,(-1)^{n+1}}{\pi n}, \quad n \ge 1. \end{aligned}$$

for solving partial differential equations such as the heat equation, one notable application of Fourier series on the square is in image compression. In particular, the jpeg image compression standard uses the two-dimensional discrete cosine transform, which is a Fourier transform using the cosine basis functions.

## IV. EULER'S FORMULA

**Euler's formula**, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric function and the comple exponential function. Euler's formula states that, for any real number *x*,

The Euler formula, sometimes also called the Euler identity

$$e^{ix} = \cos x + i\sin x,$$

where *i* is the imaginary unit. Note that Euler's polyhedral formula is sometimes also called the Euler formula, as is the Euler curvature formula. The equivalent expression

$$i\,x = \ln(\cos x + i\sin x)$$

had previously been published by Cotes (1714).

The special case of the formula with $x = \pi$ gives the beautiful identity

$$e^{i\pi} + 1 = 0,$$

an equation connecting the fundamental numbers *i*, pi, *e*, 1, and 0 (zero), the fundamental operations $+$, $\times$, and exponentiation, the most important relation $=$, and nothing else. Gauss is reported to have commented that if this formula was not immediately obvious, the reader would never be a first-class mathematician (Derbyshire 2004, p. 202).

The Euler formula can be demonstrated using a series expansion

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} \\ &= \cos x + i\sin x. \end{aligned}$$

It can also be demonstrated using a complex integral. Let

$$\begin{aligned} z &\equiv \cos\theta + i\sin\theta \\ dz &= (-\sin\theta + i\cos\theta)\,d\theta \\ &= i\,(\cos\theta + i\sin\theta)\,d\theta \\ &= i\,z\,d\theta \end{aligned}$$

$$\int \frac{dz}{z} = \int i\,d\theta$$

$$\ln z = i\,\theta,$$

so

$$\begin{aligned} z &= e^{i\theta} \\ &\equiv \cos\theta + i\sin\theta. \end{aligned}$$

A mathematical joke asks, "How many mathematicians does it take to change a light bulb?" and answers "$-e^{i\pi}$" (which, of course, equals 1)[3]

## V. CONVOLUTION THEOREM

One of the most important concepts in Fourier theory, and in crystallography, is that of a convolution. Convolutions arise in many guises, as will be shown below. Because of a mathematical property of the Fourier transform, referred to as the convolution theorem, it is convenient to carry out calculations involving convolutions.

Mathematically, a convolution is defined as the integral over all space of one function at x times another function at u-x. The integration is taken over the variable x (which may be a 1D or 3D variable), typically from minus infinity to infinity over all the dimensions. So the convolution is a function of a new variable u, as shown in the following equations. The cross in a circle is used to indicate the convolution operation. it doesn't matter which function you take first, *i.e.* the convolution operation is commutative. We'll prove that below, but you should think about this in terms of the illustration below.

$$\begin{aligned} \mathrm{C}(\mathbf{u}) = \mathrm{f}(\mathbf{x}) \otimes \mathrm{g}(\mathbf{x}) &= \int_{space} \mathrm{f}(\mathbf{x})\,\mathrm{g}(\mathbf{u}-\mathbf{x})\,d\mathbf{x} \\ = \mathrm{g}(\mathbf{x}) \otimes \mathrm{f}(\mathbf{x}) &= \int_{space} \mathrm{g}(\mathbf{x})\,\mathrm{f}(\mathbf{u}-\mathbf{x})\,d\mathbf{x} \end{aligned}$$

This illustration shows how you can think about the convolution, as giving a weighted sum of shifted copies of one function: the weights are given by the function value of the second function at the shift vector. The top pair of graphs shows the original functions. The next three pairs of graphs show (on the left) the function g shifted by various values of x and, on the right, that shifted function g multiplied by f at the value of x.

- **Proof of the convolution theorem**

. To prove the convolution theorem, in one of its statements, we start by taking the Fourier transform of a convolution. What we want to show is that this is equivalent to the product of the two individual Fourier transforms. Note, in the equation below, that the convolution integral is taken over the variable *x* to give a function of *u*. The Fourier transform then involves an integral over the variable *u*.

$$\mathbf{T}\big(\mathrm{f}(x)\otimes \mathrm{g}(x)\big)=\mathbf{T}\left(\int_{-\infty}^{\infty}\mathrm{f}(x)\,\mathrm{g}(u-x)\,dx\right)$$

$$=\int^{\infty}\int^{\infty}\mathrm{f}(x)\,\mathrm{g}(u-x)\,dx\,\exp(2\pi isu)\,du$$

$$\mathbf{T}\big(\mathrm{f}(x)\otimes \mathrm{g}(x)\big)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{f}(x)\,\mathrm{g}(w)\exp\big[2\pi is(x+w)\big]\,dx\,dw$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{f}(x)\exp(2\pi isx)\,\mathrm{g}(w)\exp(2\pi isw)\,dx\,dw$$

Now we substitute a new variable *w* for *u-x*. As above, the infinite integration limits don't change. Then we expand the exponential of a sum into the product of exponentials and rearrange to bring together expressions in *x* and expressions in *w*.

Expressions in *x* can be taken out of the integral over *w* so that we have two separate integrals, one over *x* with no terms containing *w* and one over *w* with no terms containing *x*.

The variables of integration can have any names we please,

$$\mathbf{T}\big(\mathrm{f}(x)\otimes \mathrm{g}(x)\big)=\int^{\infty}\mathrm{f}(x)\exp(2\pi isx)\,dx\int^{\infty}\mathrm{g}(w)\exp(2\pi isw)\,dw$$

$$\mathbf{T}\big(\mathrm{f}(x)\otimes \mathrm{g}(x)\big)=\int_{-\infty}^{\infty}\mathrm{f}(x)\exp(2\pi isx)\,dx\int_{-\infty}^{\infty}\mathrm{g}(x)\exp(2\pi isx)\,dx$$

$$=\mathbf{T}\big(\mathrm{f}(x)\big)\,\mathbf{T}\big(\mathrm{g}(x)\big)$$

so we can now replace *w* with *x*, and we have the result we wanted to prove.

If you look through the derivation above, you will see that we could have used a minus sign in the exponential when taking the original Fourier transform, and then the two Fourier transforms at the end would also contain minus signs in the exponentials. In other words, the convolution theorem applies to both the forward and reverse Fourier transforms. This is not surprising, since the two directions of Fourier transform are essentially identical [4]

## VI. CONCLUSION

The Fourier series is most important to  knw because it has the quite broad applications in engineering field and in communication field as well. We can know about it by going through the history.

**Reference**


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[2] http://en.wikipedia.org/wiki/Fourier_series

[3] http://mathworld.wolfram.com/EulerFormula.html

[4] http://www.structmed.cimr.cam.ac.uk/Course/Convolution/convolution.html