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GROUP CONCEPTS, RING CONCEPTS AND GROUP HOMOMORPHISM OF DOUBLY STOCHASTIC MATRIX

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Abstract- **Defining the group concept, ring concept and also group homomorphism of doubly stochastic matrix. The basic concepts and theorems of the above are introduced with examples.**

Index Terms- **Doubly stochastic group, doubly stochastic group homomorphism and doubly stochastic ring.**

AMS Classifications: 15A51, 15B99

DEFINITION: 1

A collection of absolute non-singular doubly stochastic matrix (G, *) is said to be a doubly stochastic group with respect to multiplication, it satisfies the following axioms.

Axiom-1: It is closure with respect to multiplication.(i.e.) A * B $\in$ G.

Axiom-2: * is associative.
(i.e.) A* (B * C) = (A * B) * C.

Axiom-3: There exists an identity matrix I in G such that A * I = I * A = A for all a $\in$ G.

Axiom-4: For each A $\in$ G there exists a matrix $A^{-1} \in$ G such that $A * A^{-1} = A^{-1} * A = I$
$\Rightarrow A^{-1}$ is the inverse of A.

DEFINITION: 2

A doubly stochastic group (G, *) is said to be doubly stochastic abelian group if the binary operation * is commutative. (i.e.) A * B = B * A $\forall$ A, B $\in$ G.

Note:

A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and B = A = $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ then

AB = $\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$ where $c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj}$

(i.e) A = $(a_{ij})_{nxn}$ and B = $(b_{ij})_{nxn}$ then AB = $(c_{ij})_{nxn}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Product of doubly stochastic matrices is a doubly stochastic matrix.

THEOREM: 1

A doubly stochastic matrix in M_3 (R) is a doubly stochastic group with respect to multiplication.

PROOF:

Axiom-1: Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ and

B = $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix} \in M_3$ (R) then

A * B =
$$\begin{pmatrix} (1-a)b+(1-b)a & ab & (1-a)(1-b) \\ (1-a)(1-b) & (1-a)b+(1-b)a & ab \\ ab & (1-a)(1-b) & (1-a)b+(1-b)a \end{pmatrix}$$
$\in M_3$ (R)

Axiom-2: Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$

B= $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix}$ and

C= $\begin{pmatrix} 0 & 1-c & c \\ c & 0 & 1-c \\ 1-c & c & 0 \end{pmatrix} \in M_3$ (R) then

A * (B * C) =
$$\begin{pmatrix} 1-(a+b+c)+(ab+bc+ca) & (a+b+c)-2(ab+bc+ca)+3abc & (ab+bc+ca)-3abc \\ (ab+bc+ca)-3abc & 1-(a+b+c)+(ab+bc+ca) & (a+b+c)-2(ab+bc+ca)+3abc \\ (a+b+c)-2(ab+bc+ca)+3abc & (ab+bc+ca)-3abc & 1-(a+b+c)+(ab+bc+ca) \end{pmatrix}$$

Similarly (A * B) * C=
$$\begin{pmatrix} 1-(a+b+c)+(ab+bc+ca) & (a+b+c)-2(ab+bc+ca)+3abc & (ab+bc+ca)-3abc \\ (ab+bc+ca)-3abc & 1-(a+b+c)+(ab+bc+ca) & (a+b+c)-2(ab+bc+ca)+3abc \\ (a+b+c)-2(ab+bc+ca)+3abc & (ab+bc+ca)-3abc & 1-(a+b+c)+(ab+bc+ca) \end{pmatrix}$$

(i.e.) A*(B*C) = (A*B)*C. Hence * is associative.

Axiom-3: Let A= $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix} \in M_3$(R), there exists an identity

I= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3$ (R) such that

A * I = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ = A and

$I * A = \begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix} = A$

(i.e.) $A * I = I * A = A \ \forall \ A \in M_3(R)$.

Axiom-4:

Let $A = \begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix} \in M_3(R)$ Using Cayley's Hamilton theorem, we get the inverse.

The characteristic equation is $|A - \lambda I| = 0. \Rightarrow \lambda^3 + (3a^2 - 3a)\lambda - (3a^2 - 3a + 1) = 0$. Its satisfies its own characteristic equation then

$A^3 + (3a^2 - 3a) A - (3a^2 - 3a + 1) = 0$.

$\Rightarrow A^{-1} = \frac{1}{3a^2-3a+1}\begin{pmatrix} a^2-a & a^2 & a^2-2a+1 \\ a^2-2a+1 & a^2-a & a^2 \\ a^2 & a^2-2a+1 & a^2-a \end{pmatrix}$

(i.e.) $A * A^{-1} = A^{-1} * A = I$.

Hence $M_3(R)$ is a doubly stochastic group with respect to the given operation multiplication.

THEOREM: 2

A doubly stochastic matrix in $M_3(R)$ is a doubly stochastic abelian group with respect to multiplication.

PROOF:

From theorem 1 ($M_3(R)$, *) is a doubly stochastic group. Now its satisfies A * B = B * A

Let $A = \begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ and

$B = \begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix} \in M_3(R)$ then

$A * B =$

$\begin{pmatrix} (1-a)b+(1-b)a & ab & (1-a)(1-b) \\ (1-a)(1-b) & (1-a)b+(1-b)a & ab \\ ab & (1-a)(1-b) & (1-a)b+(1-b)a \end{pmatrix}$

$B * A =$

$\begin{pmatrix} (1-a)b+(1-b)a & ab & (1-a)(1-b) \\ (1-a)(1-b) & (1-a)b+(1-b)a & ab \\ ab & (1-a)(1-b) & (1-a)b+(1-b)a \end{pmatrix}$

$\Rightarrow A * B = B * A \ \forall \ A, B \in M_3(R)$.

Hence $M_3(R)$ is a doubly stochastic abelian group with respect to the given operation multiplication.

EXAMPLE: 1

Let a = 1/2 and b = 1/3 then

$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

(i) $A * B = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \\ 1/6 & 2/6 & 3/6 \end{pmatrix} \in M_3(R)$

(ii) Let c = 1/4 then $C = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}$

$A * (B * C) = \begin{pmatrix} 7/24 & 11/24 & 6/24 \\ 6/24 & 7/24 & 11/24 \\ 11/24 & 6/24 & 7/24 \end{pmatrix}$ and

$(A * B) * C = \begin{pmatrix} 7/24 & 11/24 & 6/24 \\ 6/24 & 7/24 & 11/24 \\ 11/24 & 6/24 & 7/24 \end{pmatrix}$

$\Rightarrow$ * is associative

(iii) There exists an identity $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then

$A * I = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = A$ and

$I * A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = A$

(iv) There exists an inverse using theorem 1,

$A^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ then

$A * A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ and

$A^{-1} * A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

(v) $A * B = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \\ 1/6 & 2/6 & 3/6 \end{pmatrix}$ and

$B * A = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \\ 1/6 & 2/6 & 3/6 \end{pmatrix}$

Hence the given doubly stochastic matrix in $M_3(R)$ is an abelian group with respect to multiplication.

DEFINITION: 3

A collection of non-singular doubly stochastic matrix (G, +) is said to be a doubly stochastic group with respect to addition, it satisfies the following properties.

Axiom-1: It is closure with respect to multiplication. (i.e.) $1/2\,(A + B) \in G$.

Axiom -2: Addition is associative. (i.e.)$1/3[A + (B + C)] = 1/3[(A + B) + C]$.

Axiom-3: There exists an identity matrix o in G such that $A + O = O + A = A$ for all $a \in G$.

Axiom-4: For each $A \in G$ there exists a matrix $A^{-1} \in G$ such that $A + A^{-1} = A^{-1} + A = O$

If $\sum_{i=1}^{n} |a_{ij}| = 1$, j = 1, 2, …n and $\sum_{j=1}^{n} |a_{ij}| = 1$, I = 1, 2, ….n

$\Rightarrow A^{-1}$ is the inverse of A

DEFINITION: 4

A doubly stochastic group (G, +) is said to be doubly stochastic abelian group if the binary operation + is commutative.

(i.e.) 1/2 [A + B] =1/2 [B + A] $\forall$ A, B $\in$ G.

THEOREM: 3

A doubly stochastic matrix in M_3 (R) is a doubly stochastic group with respect to addition.

PROOF:

Axiom-1: Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ and

B = $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix}$ $\in$ M_3 (R) then

A + B = $\begin{pmatrix} 0 & 2-(a+b) & (a+b) \\ (a+b) & 0 & 2-(a+b) \\ 2-(a+b) & (a+b) & 0 \end{pmatrix}$

1/2[A+B] =

$\begin{pmatrix} 0 & 1-(a+b) & (a+b) \\ (a+b) & 0 & 1-(a+b) \\ 1-(a+b) & (a+b) & 0 \end{pmatrix}$ $\in$ M_3 (R)

Axiom-2: Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ and

B = $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix}$

C = $\begin{pmatrix} 0 & 1-c & c \\ c & 0 & 1-c \\ 1-c & c & 0 \end{pmatrix}$ $\in$ M_3 (R) then

1/3[A + (B + C)] =

$\begin{pmatrix} 0 & 1-(a+b) & (a+b) \\ (a+b) & 0 & 1-(a+b) \\ 1-(a+b) & (a+b) & 0 \end{pmatrix}$ ……..(1)

Similarly 1/3 [(A + B) + C] =

$\begin{pmatrix} 0 & 1-(a+b) & (a+b) \\ (a+b) & 0 & 1-(a+b) \\ 1-(a+b) & (a+b) & 0 \end{pmatrix}$ …….(2)

$\Rightarrow$ 1/3[A + (B +C)] =1/3[(A +B) + C].

Axiom-3: Let A= $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ $\in$$M_3$(R), there exists an identity

O= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\in$ M_3 (R) such that

A + O = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ = A and

O + A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ = A

$\Rightarrow$ A + O = O + A = A $\forall$ A $\in$ M_3 (R).

Axiom-4: Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ $\in$ M_3 (R) and their exists an identity O in M_3 (R) then the additive inverse of A is

A^{-1} = $\begin{pmatrix} 0 & -1+a & -a \\ -a & 0 & -1+a \\ -1+a & -a & 0 \end{pmatrix}$ $\Rightarrow$A + A^{-1} = A^{-1} + A = O.

Hence M_3 (R) is a doubly stochastic group with respect to the given operation addition.

THEOREM: 4

A doubly stochastic matrix in M_3 (R) is an doubly stochastic abelian group with respect to addition.

PROOF:

From the theorem 1 (M_3(R), +) is a group. Now its satisfies 1/2 [A + B] = ½ [B + A]

Lat A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$ and

B = $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix}$ then

1/2 [A + B] =

$\begin{pmatrix} 0 & 1-(a+b) & (a+b) \\ (a+b) & 0 & 1-(a+b) \\ 1-(a+b) & (a+b) & 0 \end{pmatrix}$

1/2 [B + A] =

$\begin{pmatrix} 0 & 1-(a+b) & (a+b) \\ (a+b) & 0 & 1-(a+b) \\ 1-(a+b) & (a+b) & 0 \end{pmatrix}$

$\Rightarrow$1/2 [A + B] =1/2 [B + A] $\forall$ A, B $\in$ M_3 (R).

Hence M_3 (R) is an doubly stochastic abelian group with respect to the given operation addition.

EXAMPLE: 2

Let a = 1/2 and b = 1/3 then

A = $\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ and B= $\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

(i) A + B = $\begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

1/2[A + B] = 1/2 $\begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

= $\begin{pmatrix} 0 & 7/12 & 5/12 \\ 5/12 & 0 & 7/12 \\ 7/12 & 5/12 & 0 \end{pmatrix}$ $\in$ M_3 (R)

(ii) Let c = 1/4 then C = $\begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}$

1/3[A + (B + C)] = 1/3 $\begin{pmatrix} 0 & 46/24 & 26/24 \\ 26/24 & 0 & 46/24 \\ 46/24 & 26/24 & 0 \end{pmatrix}$

= $\begin{pmatrix} 0 & 46/72 & 26/72 \\ 26/72 & 0 & 46/72 \\ 46/72 & 26/72 & 0 \end{pmatrix}$

$1/3\,[(A + B) + C] = 1/3 \begin{pmatrix} 0 & 46/24 & 26/24 \\ 26/24 & 0 & 46/24 \\ 46/24 & 26/24 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 46/72 & 26/72 \\ 26/72 & 0 & 46/72 \\ 46/72 & 26/72 & 0 \end{pmatrix}$

$\Rightarrow$ addition is associative

(iii) There exists an identity $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then

$A + O = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = A$ and

$O + A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = A$

(iv) There exists an inverse,

$A^{-1} = \begin{pmatrix} 0 & -1/2 & -1/2 \\ -1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 \end{pmatrix}$ then

$A + A^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$ and

$A^{-1} + A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$

(v) $A + B = \begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

$1/2[A + B] = 1/2 \begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 7/12 & 5/12 \\ 5/12 & 0 & 7/12 \\ 7/12 & 5/12 & 0 \end{pmatrix}$

$B + A = \begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

$1/2[B + A] = 1/2 \begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 7/12 & 5/12 \\ 5/12 & 0 & 7/12 \\ 7/12 & 5/12 & 0 \end{pmatrix}$

Hence the given doubly stochastic matrices in $M_3(R)$ is an abelian group with respect to addition.

DEFINITION: 5

A homomorphism of a doubly stochastic group G in to G' is a map $f : G \to G'$ is defined by $f(a) = a^2$ such that f(ab) = f(a).f(b) for all a, $b \in G$ with respect to multiplication.

THEOREM: 5

A doubly stochastic group G into G' is a doubly stochastic group homomorphism with respect to multiplication such that f(ab) = f(a).f(b) for all a, $b \in G$ where $f(a) = A^2$ and $f(b) = B^2$

PROOF:

$f(ab) = (AB)^2 = (AB)\,(AB) = A(BA)B$
$= A(AB)B$ where $AB = BA$
$= A^2\, B^2 = f(a)\, f(b)$

EXAMPLE: 3

$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

and $ab = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \\ 1/6 & 2/6 & 3/6 \end{pmatrix}$

then $f(a) = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

$f(b) = \begin{pmatrix} 4/9 & 1/9 & 4/9 \\ 4/9 & 4/9 & 1/9 \\ 1/9 & 4/9 & 4/9 \end{pmatrix}$

$f(ab) = \begin{pmatrix} 13/36 & 10/36 & 13/36 \\ 13/36 & 13/36 & 10/36 \\ 10/36 & 13/36 & 13/36 \end{pmatrix}$

DEFINITION: 6

A homomorphism of a doubly stochastic group G in to G' is a map $f : G \to 1/2G'$ is defined by $f(a) = A/2$ and $f(b) = B/2$ such that $f(a+b)=f(a) + f(b)$, for all a, $b \in G$ with respect to addition.

THEOREM: 6

A doubly stochastic group G into G' is a group homomorphism with respect to addition such that $f(a+b)= f(a)+f(b)$, for all a, $b \in G$ where $f(a) = A/2$ and $f(b) = B/2$

PROOF:

$f(a+b) = \frac{A+B}{2} = \frac{A}{2} + \frac{B}{2} = f(a) + f(b)$

EXAMPLE: 4

$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

and $A + B = \begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}$

Then $f(a) = \begin{pmatrix} 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 \end{pmatrix}$

$f(b) = \begin{pmatrix} 0 & 2/6 & 1/6 \\ 1/6 & 0 & 2/6 \\ 2/6 & 1/6 & 0 \end{pmatrix}$

f(a + b) = $\begin{pmatrix} 0 & 7/12 & 5/12 \\ 5/12 & 0 & 7/12 \\ 7/12 & 5/12 & 0 \end{pmatrix}$

DEFINITION: 7

A collection of non-empty and non-singular doubly stochastic matrix R together with two binary operations denoted by "+" and "." are addition and multiplication which satisfy the following axioms is called a doubly stochastic Ring.

Axiom -1: (R, +) is an abelian group.

Axiom -2: "." is associative binary operation on R.

Axiom -3: $\frac{1}{2}$ **[A** . (B + C)] = $\frac{1}{2}$ [A . B + A . C] and

$\frac{1}{2}$ **[**(A + B) . C] = $\frac{1}{2}$ **[** A . C + B . C]

for all A, B, C ∈ R.

THEOREM: 7

A doubly stochastic matrix in M_3 (R) is a doubly stochastic ring with respect to addition and multiplication.

PROOF:

Axiom-1: We know that M_3 (R) is a doubly stochastic abelian group with respect toaddition from theorem 3 and 4.

Axiom-2: It is also satisfies the associative property with respect to multiplication from theorem 1.

Axiom -3:

Let A = $\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix}$

B= $\begin{pmatrix} 0 & 1-b & b \\ b & 0 & 1-b \\ 1-b & b & 0 \end{pmatrix}$

and C= $\begin{pmatrix} 0 & 1-c & c \\ c & 0 & 1-c \\ 1-c & c & 0 \end{pmatrix}$ ∈ M_3 (R) then

A . (B + C) =

$\begin{pmatrix} 0 & 1-a & a \\ a & 0 & 1-a \\ 1-a & a & 0 \end{pmatrix} . \begin{pmatrix} 0 & 2-(b+c) & (b+c) \\ (b+c) & 0 & 2-(b+c) \\ 2-(b+c) & (b+c) & 0 \end{pmatrix} =$

$\begin{pmatrix} 2a+b+c-2ab-2ac & ab+ac & 2-2a-b-c+ab+ac \\ 2-2a-b-c+ab+ac & 2a+b+c-2ab-2ac & ab+ac \\ ab+ac & 2-2a-b-c+ab+ac & 2a+b+c-2ab-2ac \end{pmatrix}$

$\frac{1}{2}$ [A.(B + C)] =

$\begin{pmatrix} a+\frac{b}{2}+\frac{c}{2}-ab-ac & \frac{ab}{2}+\frac{ac}{2} & 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} \\ 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} & a+\frac{b}{2}+\frac{c}{2}-ab-ac & \frac{ab}{2}+\frac{ac}{2} \\ \frac{ab}{2}+\frac{ac}{2} & 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} & a+\frac{b}{2}+\frac{c}{2}-ab-ac \end{pmatrix}$

A . B+A . C =

$\begin{pmatrix} a+b-2ab & ab & 1-a-b+ab \\ 1-a-b+ab & a+b-2ab & ab \\ ab & 1-a-b+ab & a+b-2ab \end{pmatrix} + \begin{pmatrix} a+c-2ac & ac & 1-a-c+ac \\ 1-a-c+ac & a+c-2ac & ac \\ ac & 1-a-c+ac & a+c-2ac \end{pmatrix}$

=

$\begin{pmatrix} 2a+b+c-2ab-2ac & ab+ac & 2-2a-b-c+ab+ac \\ 2-2a-b-c+ab+ac & 2a+b+c-2ab-2ac & ab+ac \\ ab+ac & 2-2a-b-c+ab+ac & 2a+b+c-2ab-2ac \end{pmatrix}$

$\frac{1}{2}$ [A.B+A.C**]** =

$\begin{pmatrix} a+\frac{b}{2}+\frac{c}{2}-ab-ac & \frac{ab}{2}+\frac{ac}{2} & 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} \\ 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} & a+\frac{b}{2}+\frac{c}{2}-ab-ac & \frac{ab}{2}+\frac{ac}{2} \\ \frac{ab}{2}+\frac{ac}{2} & 1-a-\frac{b}{2}-\frac{c}{2}+\frac{ab}{2}+\frac{ac}{2} & a+\frac{b}{2}+\frac{c}{2}-ab-ac \end{pmatrix}$

⇒ $\frac{1}{2}$ [A . (B + C)] = $\frac{1}{2}$ [A . B + A . C] and also

(A + B) . C = $\begin{pmatrix} 0 & 2-(a+b) & (a+b) \\ (a+b) & 0 & 2-(a+b) \\ 2-(a+b) & (a+b) & 0 \end{pmatrix}.$

$\begin{pmatrix} 0 & 1-c & c \\ c & 0 & 1-c \\ 1-c & c & 0 \end{pmatrix}$

=

$\begin{pmatrix} 2c+a+b-2ac-2bc & ac+bc & 2-2c-a-b+ac+bc \\ 2-2c-a-b+ac+bc & 2c+a+b-2ac-2bc & ac+bc \\ ac+bc & 2-2c-a-b+ac+bc & 2c+a+b-2ac-2bc \end{pmatrix}$

$\frac{1}{2}$ [(A + B).C] =

$\begin{pmatrix} c+\frac{a}{2}+\frac{b}{2}-ac-bc & \frac{ac}{2}+\frac{bc}{2} & 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} \\ 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} & c+\frac{a}{2}+\frac{b}{2}-ac-bc & \frac{ac}{2}+\frac{bc}{2} \\ \frac{ac}{2}+\frac{bc}{2} & 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} & c+\frac{a}{2}+\frac{b}{2}-ac-bc \end{pmatrix}$

A . C + B . C =

$\begin{pmatrix} a+c-2ac & ac & 1-a-c+ac \\ 1-a-c+ac & a+c-2ac & ac \\ ac & 1-a-c+ac & a+c-2ac \end{pmatrix} + \begin{pmatrix} b+c-2bc & bc & 1-b-c+bc \\ 1-b-c+bc & b+c-2bc & bc \\ bc & 1-b-c+bc & b+c-2bc \end{pmatrix} =$

$\begin{pmatrix} 2c+a+b-2ac-2bc & ac+bc & 2-2c-a-b+ac+bc \\ 2-2c-a-b+ac+bc & 2c+a+b-2ac-2bc & ac+bc \\ ac+bc & 2-2c-a-b+ac+bc & 2c+a+b-2ac-2bc \end{pmatrix}$

$\frac{1}{2}$[A.C+B.C]=

$\begin{pmatrix} c+\frac{a}{2}+\frac{b}{2}-ac-bc & \frac{ac}{2}+\frac{bc}{2} & 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} \\ 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} & c+\frac{a}{2}+\frac{b}{2}-ac-bc & \frac{ac}{2}+\frac{bc}{2} \\ \frac{ac}{2}+\frac{bc}{2} & 1-c-\frac{a}{2}-\frac{b}{2}+\frac{ac}{2}+\frac{bc}{2} & c+\frac{a}{2}+\frac{b}{2}-ac-bc \end{pmatrix}$

⇒ $\frac{1}{2}$ [(A + B) . C] = $\frac{1}{2}$ [A . C + B . C]

Hence a doubly stochastic matrix in M_3 (R) is a doubly stochastic ring with respect to addition and multiplication.

EXAMPLE: 5

Let a = 1/2 , b = 1/3 and c = 1/4 then

A = $\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ B = $\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$

C = $\begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}$

From example 2, the given doubly stochastic matrices in M_3(R) is an abelian group with respect to addition.

Next from example 1, the given doubly stochastic matrix in M_3(R) is an abelian group with respect to multiplication.

Next we will show that

A . (B + C) =
$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 17/12 & 7/12 \\ 7/12 & 0 & 17/12 \\ 17/12 & 7/12 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 24/24 & 7/24 & 17/24 \\ 17/24 & 24/24 & 7/24 \\ 7/24 & 17/24 & 24/24 \end{pmatrix}$$
$$\frac{1}{2}[A . (B + C)] = \begin{pmatrix} 24/48 & 7/48 & 17/48 \\ 17/48 & 24/48 & 7/48 \\ 7/48 & 17/48 & 24/48 \end{pmatrix}$$
A . B + A . C =
$$\begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \\ 1/6 & 2/6 & 3/6 \end{pmatrix}+\begin{pmatrix} 4/8 & 1/8 & 3/8 \\ 3/8 & 4/8 & 1/8 \\ 1/8 & 3/8 & 4/8 \end{pmatrix}$$
$$= \begin{pmatrix} 24/24 & 7/24 & 17/24 \\ 17/24 & 24/24 & 7/24 \\ 7/24 & 17/24 & 24/24 \end{pmatrix}$$
$$\frac{1}{2}[A . B + A . C] = \begin{pmatrix} 24/48 & 7/48 & 17/48 \\ 17/48 & 24/48 & 7/48 \\ 7/48 & 17/48 & 24/48 \end{pmatrix}$$
$\Rightarrow \frac{1}{2}[A . (B + C)] = \frac{1}{2}[A . B + A . C]$ and

(A + B) . C =
$$\begin{pmatrix} 0 & 7/6 & 5/6 \\ 5/6 & 0 & 7/6 \\ 7/6 & 5/6 & 0 \end{pmatrix}.\begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 22/24 & 5/24 & 21/24 \\ 21/24 & 22/24 & 5/24 \\ 5/24 & 21/24 & 22/24 \end{pmatrix}$$
$$\frac{1}{2}[(A + B) . C] = \begin{pmatrix} 22/48 & 5/48 & 21/48 \\ 21/48 & 22/48 & 5/48 \\ 5/48 & 21/48 & 22/48 \end{pmatrix}$$
A . C + B . C
$$= \begin{pmatrix} 4/8 & 1/8 & 3/8 \\ 3/8 & 4/8 & 1/8 \\ 1/8 & 3/8 & 4/8 \end{pmatrix} + \begin{pmatrix} 5/12 & 1/12 & 6/12 \\ 6/12 & 5/12 & 1/12 \\ 1/12 & 6/12 & 5/12 \end{pmatrix}$$
$$= \begin{pmatrix} 22/24 & 5/24 & 21/24 \\ 21/24 & 22/24 & 5/24 \\ 5/24 & 21/24 & 22/24 \end{pmatrix}$$
$$\frac{1}{2}[A . C + B . C] = \begin{pmatrix} 22/48 & 5/48 & 21/48 \\ 21/48 & 22/48 & 5/48 \\ 5/48 & 21/48 & 22/48 \end{pmatrix}$$
$\Rightarrow \quad \frac{1}{2}[(A + B) . C] = \frac{1}{2}[A . C + B . C]$

Hence the given doubly stochastic matrices in $M_3(R)$ is a doubly stochastic ring with respect to addition and multiplication.

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