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Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces

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***Abstract*— In this paper we extend some results of fixed points for asymptotically regular mappings on a complete G-metric spaces by using new approach.**

Index Terms*— *Asymptotically Regular Mapping, Fixed Point, G-Cauchy sequence, G –metric space.

I. INTRODUCTION

Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of self maps of metric spaces. The concept of asymptotically regular at a point in a space was first introduced by Browder and Petryshyn and the concept of G – metric space was introduced by Z.Mustafa and B.Sims. In the present paper we have proved some results in Complete G-metric space under asymptotic regularity with some contractive condition

II. DEFINITION AND PRELIMINARIES

Definition: 2.1

Let X be a non empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties.

(1) $G(x, y, z) = 0$ if $x = y = z$.
(2) $G(x, x, y) > 0$; for all $x, y \in X$ with $x \neq y$.
(3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z with $y \neq z$
(4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots$ (Symmetry in all three variables)
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, $a \in X$ (rectangle inequality)

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition: 2.2

Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and one say that the sequence $\{x_n\}$ is G-convergent to x.

Definition: 2.3

Let (X, G) be a G-metric space and let $\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_\ell) < \epsilon$, for all $n, m, \ell \geq N$. i.e $G(x_n, x_m, x_\ell) \to 0$, as $n, m, \ell \to \infty$.

Definition: 2.4

A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition: 2.5

A mapping $T:X\to X$ of a symmetric G-metric space (X, G) into itself is said to be asymptotically regular at a point $x \in X$,

if $\lim_{n\to\infty} G(T^{n+1}x, T^nx, T^nx) = \lim_{n\to\infty} G(T^nx, T^{n+1}x, T^{n+1}x) = 0.$

Where T^nx is n^{th} iterate of T at $x \in X$

Example: 2.6

Let R be the set of all real numbers.

Define $G: R \times R \times R \to R^+$ by

$G(x, y, z) = |x-y| + |x-z| + |y-z|$, for all $x, y, z \in R$. then (R, G) is symmetric G-metric space.

Let T be self-mapping on R with $Tx = x/2$ then

$$G(T^{n+1}x, T^nx, T^nx) = \left|\frac{x}{2^{n+1}} - \frac{x}{2^n}\right| + \left|\frac{x}{2^{n+1}} - \frac{x}{2^n}\right| + \left|\frac{x}{2^n} - \frac{x}{2^n}\right|$$
$$= 2x\left|\frac{1}{2^{n+1}} - \frac{1}{2^n}\right| = \frac{2x}{2^n}\left|\frac{1}{2} - 1\right| = \frac{x}{2^n}$$
$$\lim_{n\to\infty} G(T^{n+1}x, T^nx, T^nx) = \lim_{n\to\infty} \frac{x}{2^n} \to 0$$

Hence T is asymptotically regular at a point $x \in R$

Proposition: 2.7

Let (X, G) be G-metric space, then the following are equivalent

(i) $\{x_n\}$ is G-convergent to x.
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Proposition: 2.8

Every G-metric space (X, G) induces a metric space (X, d_G) defined by

$d_G(x, y) = G(x, y, y) + G(y, x, x)\ \forall\ x, y \in X$

Proof.

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x) \geq G(x, x, x) = 0$ (by rectangular inequality)

(ii) $d_G(x, x) = G(x, x, x) + G(x, x, x) = 0$. (i.e $d_G(x, y) = 0$ iff $x = y$.)

(iii) $d_G(x, y) = G(x, y, y) + G(y, x, x) = G(y, x, x) + G(x, y, y) = d_G(y, x)$

(iv) $d_G(x, y) = G(x, y, y) + G(y, x, x)$

$\leq G(x, z, z) + G(z, y, y) + G(y, z, z) + G(z, x, x)$

$\leq [G(x, z, z) + G(z, x, x)] + [G(z, y, y) + G(y, z, z)]$

$\leq d_G(x, z) + d_G(z, y)$

Hence (X, G) induces a metric space (X, d_G)

III. THE MAIN RESULT

Theorem: 3.1

Let (X, G) be a complete symmetric G-metric space and T: X→ X be a mapping such that the following condition is satisfied

$$G(Tx, Ty, Ty) \leq aG(x, y, y)+ b[G(x, Tx, Tx) + G(y, Ty, Ty)] + c[G(x, Ty, Ty) + G(y, Tx, Tx)] + d[G(x, Tx, Ty) + G(y, Ty, Tx)] \ldots\ldots(1)$$

for all x, y in X where $0 \leq 2a + b + 3c + 3d < 1$; $0 \leq a, b, c, d < 1$ then T has a unique fixed point in X, if T is asymptotically regular at some point in X.

Proof.

Consider the sequence $\{T^n x\}$ and assume that T is asymptotically regular at some point $x_0 \in X$.

for $n, m \geq 1$,

$G(T^n x_0, T^m x_0, T^m x_0)$

$\leq a\, G(T^{n-1}x_0, T^{m-1}x_0, T^{m-1}x_0)$
$+ b[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^{m-1}x_0, T^m x_0, T^m x_0)]$
$+ c[G(T^{n-1}x_0, T^m x_0, T^m x_0) + G(T^{m-1}x_0, T^m x_0, T^n x_0)]$
$+ d[G(T^{n-1}x_0, T^n x_0, T^m x_0) + G(T^{m-1}x_0, T^m x_0, T^n x_0)]$

$\leq a\, [G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^{m-1}x_0, T^{m-1}x_0, T^m x_0) + G(T^m x_0, T^m x_0, T^n x_0)]$
$+ b[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^{m-1}x_0, T^m x_0, T^m x_0)]$
$+ c[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^n x_0, T^m x_0, T^m x_0)$
$+ G(T^{m-1}x_0, T^m x_0, T^m x_0) + G(T^m x_0, T^n x_0, T^n x_0)]$
$+ d[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^n x_0, T^n x_0, T^m x_0)$
$+ G(T^{m-1}x_0, T^m x_0, T^m x_0) + G(T^m x_0, T^m x_0, T^n x_0)]$

i.e $(1 - a - 2c - 2d)\, G(T^n x_0, T^m x_0, T^m x_0)$
$\leq (a + b + c + d)[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^{m-1}x_0, T^m x_0, T^m x_0)]$

i.e $G(T^n x_0, T^m x_0, T^m x_0)$
$\leq \frac{(a+b+c+d)}{(1-a-2c-2d)}\, [G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(T^{m-1}x_0, T^m x_0, T^m x_0)]$

Taking limit as $m, n \to \infty$ such that

$$\lim_{m,n\to\infty} G(T^n x_0, T^m x_0, T^m x_0) \to 0 \ldots\ldots\ldots\ldots\ldots(2)$$

Since T is asymptotically regular at x_0.

i.e $G(T^{n-1}x_0, T^n x_0, T^n x_0) \to 0$ as $n \to \infty$,
$G(T^{m-1}x_0, T^m x_0, T^m x_0) \to 0$ as $m \to \infty$.

Now

$G(T^n x_0, T^m x_0, T^\ell x_0)$
$\leq G(T^n x_0, T^m x_0, T^m x_0) + G(T^m x_0, T^m x_0, T^\ell x_0)$

Taking limit as $n, m, \ell \to \infty$.

$G(T^n x_0, T^m x_0, T^\ell x_0) \to 0$ Refer to (2)

So $T^n x_0$ is a G-cauchy sequence.

Since (X,G) is complete there exist a point $u \in X$ such that $u = \lim_{n\to\infty} T^n x_0$

Suppose that u is not a fixed point of T ($Tu \neq u$) then by condition (1) and rectangular inequality we obtain

$G(u, Tu, Tu) \leq G(u, T^n x_0, T^n x_0) + G(T^n x_0, Tu, Tu)$
$\leq G(u, T^n x_0, T^n x_0) + a\, G(T^{n-1}x_0, u, u)$
$+ b[G(T^{n-1}x_0, T^n x_0, T^n x_0) + G(u, Tu, Tu)]$
$+ c[G(T^{n-1}x_0, Tu, Tu) + G(u, T^n x_0, T^n x_0)]$
$+ d[G(T^{n-1}x_0, T^n x_0, Tu) + G(u, T^n x_0, Tu)]$

Taking the limit as $n \to \infty$ we obtain

$G(u, Tu, Tu) < (b + c + 2d)\, G(u, Tu, Tu)$

This contradiction implies that $Tu = u$.

Hence u is a fixed point of T.

To prove the uniqueness of fixed point, suppose T has second fixed point v in X, then by (1)

$G(u, v, v) = G(Tu, Tv, Tv)$
$\leq aG(u, v, v) + b[G(u, Tu, Tu) + G(v, Tv, Tv)$
$+ c[G(u, Tv, Tv) + G(v, Tu, Tu)]$
$+d[G(u, Tu, Tv) + G(v, Tu, Tv)]$
$\leq (a + 2c + 2d)G(u, v, v)$

This contradiction implies that $u = v$.

Thus T has unique fixed point.

This completes the proof of Theorem 3.1

Theorem: 3.2

Let (X, G) be a complete symmetric G-metric space and T: X→ X be a mapping such that the following condition is satisfied

$G(Tx, Ty, Ty)$

≤ aG(x, y, y) + b[G(x, Tx, Tx) + G(y, Ty, Ty)]
+ c[G(x, Ty, Ty) + G(y, Tx, Tx)]
+ d[G(x, Tx, Ty) + G(y, Ty, Tx)]…(1)
for all x, y in X where $0 \leq 2a + b + 3c + 3d < 1$; $0 \leq a, b, c, d < 1$. if T is asymptotically regular at some point in x in X and sequence of iterates $\{T^n x\}$ has a sub-sequence converging to a point z ∈ X, then z is a unique fixed point of T and $\{T^n x\}$ are also converges to z.

Proof:

Let T be asymptotically regular at x ∈ X and consider the sequence $\{T^n x\}$.

We shall assume that $\lim_{k \to \infty} T^{n_k} x = z$ and $Tz \neq z$.

Then by condition (1) and rectangular inequality we obtain

$$G(z, Tz, Tz) \leq G(z, T^{n_k}x, T^{n_k}x) + G(T^{n_k}x, Tz, Tz)$$
$$\leq G(z, T^{n_k}x, T^{n_k}x) + aG(T^{n_k-1}x, z, z)$$
$$+ b[G(T^{n_k-1}x, T^{n_k}x, T^{n_k}x) + G(z, Tz, Tz)]$$
$$+c[G(T^{n_k-1}x, Tz, Tz) + G(z, T^{n_k}x, T^{n_k}x)]$$
$$+d[G(T^{n_k-1}x, T^{n_k}x, Tz) + G(z, T^{n_k}x, Tz)]$$
$$\leq G(z, T^{n_k}x, T^{n_k}x) + aG(T^{n_k-1}x, z, z)$$
$$+b[G(T^{n_k-1}x, T^{n_k}x, T^{n_k}x) + G(z, Tz, Tz)]$$
$$+c[G(T^{n_k-1}x, T^{n_k}x, T^{n_k}x) + G(T^{n_k}x, Tz, Tz) + G(z, T^{n_k}x, T^{n_k}x)]$$
$$+d[G(T^{n_k-1}x, T^{n_k}x, T^{n_k}x) + G(T^{n_k}x, T^{n_k}x, z) + G(z, T^{n_k}x, Tz)]$$

Taking limit as $k \to \infty$ we obtain

$$G(z, Tz, Tz) \leq (b + c + d)G(z, Tz, Tz)$$

This contradiction implies that Tz = z.

By Theorem: 3.1, z is the unique fixed point by using condition (1) and rectangular inequality we obtain.

To prove $\{T^n x\}$ converges to z

Let $G(z, T^n x, T^n x) \leq G(z, Tz, Tz) + G(Tz, T^n x, T^n x)$
$$\leq G(z, Tz, Tz) + aG(z, T^{n-1}x, T^{n-1}x)$$
$$+ b[G(z, Tz, Tz) + G(T^{n-1}x, T^n x, T^n x)]$$
$$+c[G(z, T^n x, T^n x) + G(T^{n-1}x, Tz, Tz)]$$
$$+d[G(z, Tz, T^n x) + G(T^{n-1}x, T^n x, Tz)]$$
$$\leq G(z, Tz, Tz)$$
$$+ a[G(z, T^n x, T^n x) + G(T^n x, T^{n-1}x, T^{n-1}x)]$$
$$+ b[G(z, Tz, Tz) + G(T^{n-1}x, T^n x, T^n x)]$$
$$+ c[G(z, T^n x, T^n x) + G(Tz, T^n x, T^n x) + G(T^n x, T^{n-1}x, T^{n-1}x)]$$
$$+d[G(z, Tz, Tz) + G(Tz, Tz, T^n x) + G(T^{n-1}x, T^n x, T^n x) + G(T^n x, T^n x, Tz)]$$

$$G(z, T^n x, T^n x) \leq \frac{(1+b+d)}{(1-a-2c-2d)} G(z, Tz, Tz) + \frac{(a+b+c+d)}{(1-a-2c-2d)} G(T^n x, T^{n-1}x, T^{n-1}x)$$

Taking limit as $n \to \infty$,

$$\lim_{n \to \infty} G(z, T^n x, T^n x) \to 0.$$

$\Rightarrow \{T^n x\}$ converges to z.

This completes the proof.

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