A Novel Bracket and the Wave Equation of a Photon

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***Abstract*- A novel kind of classical bracket of classical observables is proposed. This bracket is used directly as a derivation* of the commutator of the quantum mechanical observables that are simply obtained by Dirac quantization of the classical observables. Light bending in the presence of a massive object in Schwarzschild's metric is considered and the above bracket is used to obtain an equation of the wave function of the photon in this situation via the Dirac quantization.**

***Index Terms*- Wave Function, Photon**

I. INTRODUCTION

A detailed description of the wave function of a photon is given in [1] and [2].

II. ESTABLISHMENT OF THE NOVEL BRACKET AND THE WAVE EQUATION OF PHOTON

Notation

$\theta =$ Azimuthal Angle

$\phi =$ Polar Angle, Also a Canonical Co-ordinate

$r =$ Radial co-ordinate in the Schwarzschild setting

$V =$ Velocity of a photon

$G =$ Universal Gravitational Constant

$M =$ Massive object in a Schwarzschild setting

$t =$ Proper time

$\lambda =$ a constant of motion

$\varepsilon =$ a constant of motion

$ds =$ line element in the Schwarzschild metric

$r_0 =$ impact parameter

$p_\phi =$ Canonical momentum

$L =$ Lagrangian

$\nu =$ Frequency of a photon

$q =$ Canonical Co-ordinate

$p =$ Canonical Momentum

$\psi =$ Eigen Wave-function of the Photon in the canonical representation

Light Bending, Classical Observables (Canonical Co-ordinates) in Schwarzschild Metric.

Considering the well-known Schwarzschild metric, specially the case of the line element lying in the equatorial plane ($\theta = \pi/2$), where r, θ and ϕ have their usual meaning

$$ds^2/V^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \qquad 1$$

We have the following relationships regarding the photon orbit locally in the presence of a massive object of mass M.

From the constant of motion

$g_{tt}\frac{dt}{ds} = \varepsilon$ we have giving

$$\frac{dt}{ds} = \frac{\varepsilon}{\left(1-\frac{2GM}{r}\right)} \qquad 2$$

Similarly from another constant of motion

$g_{\phi\phi}\frac{d\phi}{ds} = \lambda$ have $-r^2\frac{d\phi}{ds} = \lambda$ giving $\frac{d\phi}{ds} = \frac{-\lambda}{r^2}$ 3

As r $\to \infty$ we note that $\frac{1}{V^2} = \varepsilon^2 - 1$ and also that $\varepsilon = \gamma$. We, therefore have

$$\frac{dr}{ds} = \left[-\frac{1}{V^2}\left(1-\frac{2GM}{r}\right) + \varepsilon^2 - \left(1-\frac{2GM}{r}\right)\frac{\lambda^2}{r^2}\right]^{\frac{1}{2}} \qquad 4$$

At $r = r_0$ we have

$$\frac{1}{V^2} = \gamma^2 - \frac{\lambda^2}{r_0^2} \qquad 5$$

From the quotient of 2 and 3 we have

$$\frac{dt}{d\phi}=\frac{\varepsilon}{\left(1-\frac{2GM}{r}\right)}\left(\frac{-r^2}{\lambda}\right) \quad 6$$

Also from the quotient of 3 and 4, we have

$$\frac{d\phi}{dr}=\frac{-\lambda}{r^2\sqrt{-\frac{1}{V^2}\left(1-\frac{2GM}{r}\right)+\varepsilon^2-\left(1-\frac{2GM}{r}\right)\frac{\lambda^2}{r^2}}}$$

6aNow considering ϕ as our canonical co-ordinate we find the canonical momentum p_ϕ given by

$p_\phi=\frac{\partial L}{\partial\dot{\phi}}$ where

$$L=-\left(\left(1-\frac{2GM}{r}\right)-\frac{1}{\left(1-\frac{2GM}{r}\right)}\left(\frac{dr}{dt}\right)^2-r^2\left(\frac{d\phi}{dt}\right)^2\right)^{\frac{1}{2}} \quad 7$$

which is gotten by using $\delta\int ds=0$

Therefore,

$$p_\phi=r^2\frac{d\phi}{ds} \quad 8$$

A Novel Bracket as the Derivation of the Commutator of the Quantum Mechanical Observables.

We propose the bracket of the form:

$$\{q,p\}_A=(q_t p_{t+dt}-p_t q_{t+dt}) \quad 9$$

where the subscript denotes the time at which the canonical co-ordinate or the canonical momentum is evaluated. Taking $dt\approx\frac{1}{\nu}$ where ν is a property of the photon, say its frequency, we note that in the limit of $dt\to 0$ we have

$$\{q,p\}_A\approx\frac{1}{\nu}(q\dot{p}-p\dot{q})$$

i.e

$$\{q,p\}_A\approx\frac{1}{\nu}\left(q\frac{dp}{dt}-p\frac{dq}{dt}\right) \quad 10$$

We further extend it to many co-ordinates in a similar fashion

$$\{q_i,p_i\}_A\approx\sum_{i=1}^{n}\frac{1}{\nu}\left(q_i\frac{dp_i}{dt}-p_i\frac{dq_i}{dt}\right) \quad 11$$

where i is an index that runs for the number of co-ordinates.

We will now comment on the motivation and the use of such a bracket. Quantizing classical dynamical systems to quantum mechanical systems involves mapping the Poisson Bracket to a Dirac Commutator by way of canonical quantization methods which incorporate the uncertainty as a function of the commutator by algebraic means. However, if we realize that a similar uncertainty as a function of the commutator can be incorporated in such a quantization map by perturbation of time in a fashion as in 9 and then evaluating the canonical co-ordinates and the canonical momenta. The inverse of this time can be conveniently taken to be of the order of the frequency of the photon whereby we do not miss any capturing of the of the wave packet nature of the photon. At this stage we can simply promote our new co-ordinates in this bracket to Quantum Mechanical observables by simply using the Dirac map.

We name this kind of bracket an Aryabhatta bracket in the honour of the ancient Indian astronomer Aryabhatta. Hence the subscript A in the notation of this bracket.

Once we promote these canonical variables namely

$q=q_A$ and $\xi\frac{1}{\nu}\frac{dp}{dt}=p_A$ 12

or $\zeta\frac{1}{\nu}\frac{dq}{dt}=q_A$ and $p=p_A$ 13

where ξ and ζ are some constants.

which we recognize as canonical variables again ready to be promoted to be quantum operators by simply doing

Note: If q_A is the co-ordinate at time t then p_A is the co-ordinate at time $t+dt$ and vice-versa.

$$\hat{q}_A=q_A=q \quad 14$$

$$\hat{p}_A=\xi\frac{h}{i\nu}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q}\right)=-\xi\frac{ih}{\nu}\frac{\partial^2}{\partial t\partial q} \quad 15$$

or

$$\hat{q}_A=q_A=\zeta\frac{1}{\nu}\frac{dq}{dt} \quad \text{and} \quad 16$$

$$\hat{p}_A=p_A=-\zeta\, ih\left(\frac{\partial}{\partial q}\right) \quad 17$$

which satisfy

$$\hat{q}_A\hat{p}_A-\hat{p}_A\hat{q}_A=1 \quad 18$$

Substituting the 14 , 15 , 16 and 17 in 18 according the note mentioned above in the above equation we get

$$q\xi\left(-\frac{ih}{\nu}\frac{\partial^2}{\partial t\partial q}\right)-\left(-ih\frac{\partial}{\partial q}\right)\zeta\left(\frac{1}{\nu}\frac{\partial q}{\partial t}\right)=1 \quad 19$$

For an eigen wave function ψ in the canonical ϕ representation that satisfies our quantum commutator we have

$$q\xi\left(-\frac{ih}{\nu}\frac{\partial^2\psi}{\partial t\partial q}\right)-\left(-ih\frac{\partial\psi}{\partial q}\right)\zeta\left(\frac{1}{\nu}\frac{\partial q}{\partial t}\right)=\psi$$

which can be rewritten as 20

$$q\xi\left(-\frac{ih}{\nu}\frac{\partial^2\psi}{\partial t\partial q}\right)-\left(-\frac{ih\zeta}{\nu}\frac{\partial\psi}{\partial t}\right)=\psi \quad 21$$

$$-ihq\xi\frac{\partial^2\psi}{\partial t\partial q}+ih\zeta\frac{\partial\psi}{\partial t}=\nu\psi \quad 22$$

or

$$\frac{h}{i}q\xi\frac{\partial^2\psi}{\partial t\partial q}-\frac{h}{i}\zeta\frac{\partial\psi}{\partial t}=\nu\psi$$

$$\boxed{q\xi\frac{\partial^2\psi}{\partial t\partial q}-\zeta\frac{\partial\psi}{\partial t}=\frac{i\nu\psi}{h}}$$

is the differential equation of the wave-function. 23

Choosing the constants ξ and ζ to be 1 we have

$$q\frac{\partial^2\psi}{\partial t\partial q}-\frac{\partial\psi}{\partial t}=\frac{i\nu\psi}{h} \quad 24$$

Wave Function of a Photon

Noting q as our canonical co-ordinate and the canonical momentum p

$$\{q,p\}_A=\frac{1}{\nu}\left(q\frac{dp}{dt}-p\frac{dq}{dt}\right)$$
$$=\frac{1}{\nu}\left(q\frac{d}{dt}\left(-ih\frac{\partial}{\partial q}\right)-\left(-ih\frac{\partial}{\partial q}\right)\frac{dq}{dt}\right) \quad 25$$
$$=\frac{ih}{\nu}\left(-q\frac{\partial^2}{\partial t\partial q}+\left(\frac{\partial}{\partial q}\right)\frac{dq}{dt}\right)$$

$$\hat{q}_A\,\hat{p}_A-\hat{p}_A\,\hat{q}_A=1=\frac{ih}{\nu}\left(-q\frac{\partial^2}{\partial t\partial q}+\left(\frac{\partial}{\partial q}\right)\frac{dq}{dt}\right)$$

From 18 26

Therefore for our wave function ψ we have

$$\boxed{q\frac{\partial^2\psi}{\partial t\partial q}-\frac{\partial\psi}{\partial t}=\frac{i\nu\psi}{h}}$$ From25 27

Decomposition of the Wave Function

We write the wave function $\psi(r,\phi,\theta)$ as

$$\psi(r,\phi,\theta)=R(r)\Phi(\phi)\Theta(\theta) \quad 27a$$

In our case, for the photon orbit lying in the equatorial plane we have,

$$\psi(r,\phi,\theta)=\psi(r,\phi)=R(r)\Phi(\phi) \quad 27b$$

Equation of R(r) component of Wave Function of the Photon

Explicitly the momentum operator in the radial and angular co-ordinates respectively is

$$p_r=r\frac{\partial}{\partial r} \quad 28$$

and $p_\phi=\phi\dfrac{1}{r}\dfrac{\partial}{\partial\phi}$ 29

(implying that p_ϕ is either discontinuous in ϕ or multiple valued).

Therefore we have for radial part we have

$$\hat{q}_A=q=\phi \quad \text{and} \quad \hat{p}_A=-ih\frac{\partial}{\partial t}\left(r\frac{\partial}{\partial r}\right)=-ih\left(\frac{\partial r}{\partial t}\frac{\partial}{\partial r}+r\frac{\partial^2}{\partial t\partial r}\right)$$

for the first term in the LHS of 18 and 30

$$\hat{p}_A=-ih\left(r\frac{\partial}{\partial r}\right) \quad \text{and} \quad \hat{q}_A=\frac{\partial\phi}{\partial t}$$ for the second term in the LHS of 18 31

Therefore we have,

$$\hat{q}_A\,\hat{p}_A-\hat{p}_A\,\hat{q}_A=\phi\left(-ih\left(\frac{\partial r}{\partial t}\frac{\partial}{\partial r}+r\frac{\partial^2}{\partial t\partial r}\right)\right)--ih\left(r\frac{\partial}{\partial r}\right)\left(\frac{\partial\phi}{\partial t}\right)=1\nu \quad 32$$

For our previously mentioned $R(r)$ but only the r component (representation) is

$$\phi\left(-ih\left(\frac{\partial r}{\partial t}\frac{\partial R(r)}{\partial r}+r\frac{\partial^2R(r)}{\partial t\partial r}\right)\right)--ih\left(r\frac{\partial R(r)}{\partial r}\right)\left(\frac{\partial\phi}{\partial t}\right)=1\nu \quad 33$$

i.e we have

$$\phi\left(\frac{\partial r}{\partial t}\frac{\partial R(r)}{\partial r}+r\frac{\partial^2R(r)}{\partial t\partial r}\right)-\left(r\frac{\partial R(r)}{\partial r}\right)\left(\frac{\partial\phi}{\partial t}\right)=-\frac{\nu}{ih} \quad 34$$

$$\phi\frac{\partial r}{\partial t}\frac{\partial R(r)}{\partial r}+\phi r\frac{\partial^2R(r)}{\partial t\partial r}-r\frac{\partial R(r)}{\partial r}\frac{\partial\phi}{\partial t}=\frac{-\nu}{ih} \quad 35$$

Since r and t are independent variables, we have

$$\boxed{\phi r\frac{\partial^2R(r)}{\partial t\partial r}-r\frac{\partial R(r)}{\partial r}\frac{\partial\phi}{\partial t}=\frac{-\nu}{ih}} \quad 36$$

for the R(r) component wave function of the photon.

Equation of R(r) component of Wave Function of the Photon in the Limit of mass M considered in the Schwarzschild setting going to zero.

The equation

$$\phi r \frac{\partial^2 R(r)}{\partial t \partial r} - r\frac{\partial R(r)}{\partial r}\frac{\partial \phi}{\partial t} = \frac{-\nu}{ih}$$

$$\phi r \frac{\partial^2 R(r)}{\partial t \partial r} - r\frac{\partial R(r)}{\partial r}\left(\frac{-\lambda}{\varepsilon r^2}\right) = \frac{-\nu}{ih}$$

Note: λ is the constant of motion; momentum. 37

as $\frac{\partial \phi}{\partial t} = \left(\frac{-\lambda}{\varepsilon r^2}\right)$ as $M \to 0$ in equation 6 37a

giving

$$\boxed{\phi r \frac{\partial^2 R(r)}{\partial t \partial r} + \frac{\partial R(r)}{\partial r}\left(\frac{\lambda}{\varepsilon r}\right) = \frac{-\nu}{ih}}$$ 37b

Equation of $\Phi(\phi)$ component of Wave Function of the Photon

$p_\phi = \phi \frac{1}{r}\frac{\partial}{\partial \phi}$ From 30

Therefore we have for angular part we have

$\hat{q}_A = q = \phi$

and

$$\hat{p}_A = -ih\frac{\partial}{\partial t}\left(\phi \frac{1}{r}\frac{\partial}{\partial \phi}\right) = -ih\left(\frac{\partial \phi}{\partial t}\frac{1}{r}\frac{\partial}{\partial \phi} + \phi\frac{\partial (r^{-1})}{\partial t}\frac{\partial}{\partial \phi} + \phi\frac{1}{r}\frac{\partial^2}{\partial t \partial \phi}\right)$$ 38

for the first term in the LHS of 18 and

$\hat{p}_A = -ih\left(\phi \frac{1}{r}\frac{\partial}{\partial \phi}\right)$ and $\hat{q}_A = \frac{\partial \phi}{\partial t}$ for the second term in the LHS of 18 39

Therefore we have,

$$\hat{q}_A\hat{p}_A - \hat{p}_A\hat{q}_A = -ih\phi\left(\frac{\partial \phi}{\partial t}\frac{1}{r}\frac{\partial}{\partial \phi} + \phi\frac{\partial (r^{-1})}{\partial t}\frac{\partial}{\partial \phi} + \phi\frac{1}{r}\frac{\partial^2}{\partial t \partial \phi}\right) - -ih\left(\phi\frac{1}{r}\frac{\partial}{\partial \phi}\right)\frac{\partial \phi}{\partial t} = 1\nu$$ 40

Since r and t are independent variables, we have

$$\hat{q}_A\hat{p}_A - \hat{p}_A\hat{q}_A = -ih\phi\left(\frac{\partial \phi}{\partial t}\frac{1}{r}\frac{\partial}{\partial \phi} + \phi\frac{1}{r}\frac{\partial^2}{\partial t \partial \phi}\right) - -ih\left(\phi\frac{1}{r}\frac{\partial}{\partial \phi}\right)\frac{\partial \phi}{\partial t} = 1\nu$$ 40a

i.e, for our previously mentioned $\Phi(\phi)$ but only the ϕ component (representation) is

$$-ih\phi\left(\frac{\partial \phi}{\partial t}\frac{1}{r}\frac{\partial \Phi(\phi)}{\partial \phi} + \phi\frac{1}{r}\frac{\partial^2 \Phi(\phi)}{\partial t \partial \phi}\right) - -ih\left(\phi\frac{1}{r}\frac{\partial \Phi(\phi)}{\partial \phi}\right)\frac{\partial \phi}{\partial t} = 1\nu$$ 41

i.e we have

$$\phi\frac{\partial \phi}{\partial t}\frac{1}{r}\frac{\partial \Phi(\phi)}{\partial \phi} + \phi\frac{1}{r}\frac{\partial^2 \Phi(\phi)}{\partial t \partial \phi} - \phi\frac{1}{r}\frac{\partial \Phi(\phi)}{\partial \phi}\frac{\partial \phi}{\partial t} = \frac{-\nu}{ih}$$ 42

i.e,

$$\boxed{\phi\frac{1}{r}\frac{\partial^2 \Phi(\phi)}{\partial t \partial \phi} = \frac{i\nu}{h}}$$ 43

for the $\Phi(\phi)$ component of wave function of the photon.

Equation of $\Phi(\phi)$ component of Wave Function of the Photon in the Limit of mass M considered in the Schwarzschild setting going to zero.

In the limit of $M \to 0$ equation 6 becomes

$$\frac{d\phi}{dr} = \frac{-\lambda}{r^2\sqrt{\varepsilon^2 - \left(\frac{\lambda^2}{r^2} + \frac{1}{V^2}\right)}}$$ 43a

$$\frac{d\phi}{dr} = \frac{-\lambda^2}{\sqrt{\left(\frac{\varepsilon^2}{\lambda^2} - \frac{1}{V^2\lambda^2}\right)r^4 - r^2}}$$

$\frac{d\phi}{dr} = \frac{A}{\sqrt{Br^4 - r^2}}$ where $A = -\lambda^2$ and $B = \frac{\varepsilon^2}{\lambda^2} - \frac{1}{V^2\lambda^2}$ 43b

Therefore,

$$\phi = \int \frac{A\,dr}{\sqrt{Br^4 - r^2}}$$ 43c

III. CONCLUSIONS

This procedure can also be extended appropriately to find the wave function of any sub-atomic particle by using its appropriate ν which is a property of the sub-atomic particle.

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