I−Convergence of Ultra filters

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Abstract- **In this paper, we have extended the idea of I−convergence of filters to the I−convergence of ultra-filters containing that filter and studied its various properties.**

I. INTRODUCTION

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [24]. Kostyrko et. al in [10] and [11] generalized the notion of statistical convergence and introduced the concept of I−convergence of real sequences which is based on the structure of the ideal I of subsets of the set of natural numbers. Mursaleen et. al [16] defined and studied the notion of ideal convergence in random 2−normed spaces and construct some interesting examples. Several works on I−convergence and statistical convergence have been done in [1], [3], [6], [7], [8], [9], [10], [11], [12], [15], [16], [17], [18], [19], [23].

The idea of I−convergence has been extended from real number space to metric space [10] and to a normed linear space [22] in recent works. Later the idea of I−convergence was extended to an arbitrary topological space by B. K. Lahiri and P. Das in [13]. It was observed that the basic properties remained preserved in topological spaces. Lahiri and Das [14] introduced the idea of I−convergence of nets in topological spaces and examined how far it affects the basic properties.

Taking the idea of [14], Jamwal et. al introduced the idea of I−convergence of filters in [6] and studied its various properties. Jamwal et. al reintroduced the idea of I−convergence of nets in topological spaces and estabilished the equivalence of I−convergences of nets and filters on topological spaces in [7]. In [8], Jamwal et. al introduced the idea of I−cluster point of filters and studied its various properties. Jamwal et. al estabilished the equivalence of I−cluster points of filters and cluster points of nets as well as the equivalence of I−cluster points of filters and nets in [9].

We start with the following definitions:

Definition 1.1 Let X be a non-empty set. Then a family $Ƒ \subset 2^X$ is called a *filter* on X if
(i) $\emptyset \notin Ƒ$,
(ii) $A, B \in Ƒ$ implies $A \cap B \in Ƒ$ and
(iii) $A \in Ƒ$, $B \supset A$ implies $B \in Ƒ$.

Definition 1.2 Let X be a non-empty set. Then a family $I \subset 2^X$ is called an *ideal* of X if
(i) $\emptyset \in I$,
(ii) $A, B \in I$ implies $A \cup B \in I$ and
(iii) $A \in I$, $B \subset A$ implies $B \in I$.

Definition 1.3 Let X be a non-empty set. Then a filter Ƒ on X is said to be *non-trivial* if $Ƒ \neq \{X\}$.

Definition 1.4 Let X be a non-empty set. Then an ideal I of X is said to be *non-trivial* if $I \neq \{\emptyset\}$ and $X \notin I$.

Note (i) $Ƒ = Ƒ(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X, called the *filter associated with the ideal* I.
(ii) $I = I(Ƒ) = \{A \subset X : X \setminus A \in Ƒ\}$ is an ideal of X, called the *ideal associated with the filter* Ƒ.
(iii) A non-trivial ideal I of X is called *admissible* if I contains all the singleton subsets of X.
Several examples of non-trivial admissible ideals have been considered in [10].
Throughout this paper, X will stand for a topological space and I = I(Ƒ) will be the ideal associated with the filter Ƒ on X.
We give a brief discussion on I−convergence and I−cluster points of filters and nets in topological spaces as given by [6], [7], [8], [9].

Definition 1.5 A filter Ƒ on X is said to be *I−convergent* to $x_0 \in X$ if for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$.
In this case, x_0 is called an *I−limit* of Ƒ and is written as $I - \lim Ƒ = x_0$.

Definition 1.6 A point $x_0 \in X$ is called an *I−cluster point* of a filter Ƒ on X if for each nbd U of x_0, {y ∈

$X : y \in U \} \notin I$. In other words, $x_0 \in X$ is called an I–cluster point of Ƒ if $U \notin I$, for each nbd U of x_0 . Equivalently, x_0 is an I–cluster point of Ƒ if for each nbd U of x_0 , $\{V \in P(X) : U \subset V\} \notin I$.

Definition 1.7 Let I be a non-trivial ideal of subsets of X. Let $\lambda : D \to X$ be a net in X, where D is a directed set. Then λ is said to be *I–convergent* to x_0 in X if for each nbd U of x_0 , $\{\lambda(c) \in X : \lambda(c) \notin U\} \in I$.

Notation In case more than one filters is involved, we use the notation I(Ƒ) to denote the ideal associated with the corresponding filter Ƒ.

Proposition 1.8 Let Ƒ be a filter on X such that I − lim Ƒ = x_0 . Then every filter Ģ on X finer than Ƒ also I–converges to x_0 , where I = I(Ƒ).

Proposition 1.9 Let Ƒ be a filter on X such that I − lim Ƒ = x_0. Then every filter Ģ on X coarser than Ƒ also I–converges to x_0, where I = I(Ƒ).

Proposition 1.10 Let M be a collection of all those filters Ģ on a space X which I(Ģ)–converge to the same point $x_0 \in X$. Then the intersection Ƒ of all the filters in M I(Ƒ)–converges to x_0.

Proposition 1.11 Let Ƒ be a filter on X and Ģ be a filter on X finer than Ƒ. Then Ƒ has x_0 as an I(Ģ)–cluster point if and only if I(Ģ) − lim Ģ = x_0.

Proposition 1.12 If X is Hausdorff, then an I–convergent filter Ƒ on X has a unique I–limit.

Proposition 1.13 If every I–convergent filter Ƒ on X has a unique I–limit, then the space X is Hausdorff.

Proposition 1.14 Let Ƒ be a filter on X and Ģ be any other filter on X finer than Ƒ. Then I(Ƒ) − lim Ģ = x_0 implies I(Ģ) − lim Ģ = x_0. But not conversely.

Proposition 1.15 Let $E \subset X$. Then $x_0 \in E$ if and only if there is a filter Ƒ on X such that E ∈ Ƒ and I − lim Ƒ = x_0.

Theorem 1.16 A filter Ƒ on X I–converges to $x_0 \in X$ if and only if every derived net λ of Ƒ converges to x_0, where I = I(Ƒ).

Theorem 1.17 A net $\lambda : D \to X$ converges to $x_0 \in X$ if and only if the derived filter Ƒ of λ I– converges to x_0, where I = I(Ƒ).

Theorem 1.18 A filter Ƒ on X I–converges to $x_0 \in X$ if and only if every derived net λ of Ƒ I–converges to x_0, where I = I(Ƒ).

Lemma 1.19 A filter Ƒ on X converges to x_0 in X if and only if every derived net λ of Ƒ I–converges to x_0, where I = I(Ƒ).

Theorem 1.20 Let $\lambda : D \to X$ be a net in X and Ƒ be a derived filter of λ. Then λ I–converges to x_0 in X if and only if the derived filter Ƒ I–converges to x_0, where I = I(Ƒ).
Theorem 1.21 A filter Ƒ I_X –converges to x in $X = \Pi_{\alpha\in\Lambda} X_\alpha$ if and only if p_α (Ƒ) $I_{X\alpha}$–converges to $p_\alpha(x)$, $\forall\, \alpha$, where $I_X = I_X$ (Ƒ) and $I_{X\alpha} = I_{X\alpha}$ (p_α(Ƒ)).

II. I-CONVERGENCE OF ULTRAFILTERS

We begin this section with the following results.

Theorem 2.1 A filter Ƒ on X I–converges to x_0 in X if and only if every ultrafilter on X containing Ƒ I–converges to x_0, where I = I(Ƒ).

Proof**.** Suppose I − lim Ƒ = x_0. Let Ģ be an ultrafilter on X containing Ƒ. Since Ģ ⊃ Ƒ, by Proposition 1.8, I − lim Ģ = x_0, where I = I(Ƒ).
Conversely, suppose that every ultrafilter Ģ on X containing the filter Ƒ I–converges to x_0, where I = I(Ƒ). By Proposition 1.14, I(Ģ) − lim Ģ = x_0.
Since I(Ģ) − lim Ģ = x_0, by Proposition 1.10 the filter ∩{Ģ : Ģ ⊃ Ƒ and Ģ I(Ģ) − converges to x_0} I(Ƒ)–converges to x_0. Evidently, the value of this intersection is Ƒ (By Proposition 7, page 61, [2]). Consequently, Ƒ also I–converges to x_0 , where I = I(Ƒ).

Lemma 2.2 If a filter Ƒ on X I(Ƒ)–converges to x_0 in X, then every ultrafilter Ģ on X containing Ƒ I(Ģ)–converges to x_0. But not conversely.

Proof. It follows from above Theorem 2.1 and Proposition 1.14.

But converse may not be true. That is, if an ultrafilter Ģ containing a filter Ƒ on X I(Ģ)−converges to $x_0 \in X$, then the filter Ƒ on X may not I(Ƒ)−converge to x_0. Consider the example:
Let X = {1, 2, 3} and τ = {∅, {2}, {2, 3}, X} be a topology on X.
Let Ƒ = {{2, 3}, X} be a filter on X. Then I(Ƒ) = {∅, {1}} is an ideal associated with Ƒ.
It is easy to see that I(Ƒ) − lim Ƒ = 3.
Let Ģ = {{2}, {1, 2}, {2, 3}, X} be an ultrafilter on X containing Ƒ. Then I(Ģ) ={∅, {1}, {3}, {1, 3}}.
We can easily see that 1, 2 and 3 are I(Ģ)−limits of Ģ. But 2 is not an I(Ƒ)−limit of Ƒ.

Lemma 2.3 Let Ƒ be a filter on X and Ģ be an ultrafilter on X containing Ƒ. Then I(Ģ) − lim Ƒ = x_0 if and only if I(Ģ) − lim Ģ = x_0.

Proof. Suppose I(Ģ) − lim Ƒ = x_0 · · · (∗). Let Ģ be an ultrafilter on X containing Ƒ. We have to show that I(Ģ) − lim Ģ = x_0. For this, let U be a nbd of x_0. We claim that {y ∈ X : y ∉ U } ∈ I(Ģ). The claim follows clearly by (∗).
Hence I(Ģ) − lim Ģ = x_0.
Converse follows clearly by Proposition 1.9.

Theorem 2.4 Let Ƒ be a filter on X and Ģ be an ultrafilter on X containing Ƒ. Then Ƒ has x_0 as an I−cluster point if and only if Ģ is I−convergent to x_0, where I = I(Ģ).

Proof. Let Ƒ be a filter on X and Ģ be an ultra-filter on X containing Ƒ. Then by Proposition 1.11, Ƒ has x_0 as an I−cluster point if and only if I − lim Ģ = x_0 , where I = I(Ģ).

Theorem 2.5 An ultrafilter Ģ I−converges to a point x_0 in X if and only if x_0 is an I−cluster point of Ģ, where I = I(Ģ).

Proof. Suppose I − lim Ģ = x_0 . Then by Proposition 1.11, I−cluster point of Ģ is x_0 . This is because Ģ is maximal and a filter finer than Ģ is Ģ itself.
Conversely, suppose x_0 is an I−cluster point of the ultrafilter Ģ on X. Then by Proposition 1.11, there is a filter Ƒ finer than Ģ such that I(Ƒ) − lim Ƒ = x_0 . But Ģ is maximal, so Ƒ = Ģ.
Hence I − lim Ģ = x_0 , where I = I(Ģ).

Theorem 2.6 X is Hausdorff if and only if every I−convergent ultrafilter Ƒ on X has a unique I−limit, where I = I(Ƒ).

Proof. Suppose X is Hausdorff. Let Ƒ be an I−convergent ultrafilter on X. Since Ƒ is a filter on X, by Proposition 1.12, Ƒ has a unique I−limit.
Conversely, suppose each I−convergent ultrafilter on X has a unique I−limit. We have to show that X is Hausdorff. Suppose X is not Hausdorff. Then by Proposition 1.13, there exists an I−convergent filter, say Ƒ on X which does not have a unique I(Ƒ)−limit. By above Theorem 2.1, there exists an I(Ƒ)−convergent ultrafilter Ģ containing Ƒ which does not have a unique I(Ƒ)−limit and so by Proposition 1.14, Ģ does not have a unique I(Ģ)−limit, which is a contradiction. Therefore, our supposition is wrong.
Hence X is Hausdorff.

Proposition 2.7 A space X is compact if and only if each ultrafilter on X is I−convergent.

Proof. First suppose X is compact. We have to show that each ultrafilter on X is I−convergent. Suppose not. Then there is an ultrafilter Ƒ on X such that Ƒ does not I−converge to any x ∈ X, where I = I(Ƒ). Then for each x in X, there is an (open) nbd Ux containing x such that $\{V \in P(X) : U_x \cap V = \emptyset\} \not\subseteq I$ …(*).
Clearly, $\{U_x : x \in X\}$ is an open cover of X. Since X is compact, the above open cover of X has a finite sub cover, say $\{Ux_i : i = 1, 2, \ldots, n\}$.
Now, $\cup^n_{i=1} Ux_i = X$ and X ∈ Ƒ
⇒ $\cup^n_{i=1} Ux_i$ ∈ Ƒ
⇒ Ux_i ∈ Ƒ, for some i
⇒ $X \setminus Ux_i \in I$, for some i, which contradicts (∗) as $Ux_i \cap (X \setminus Ux_i) = \emptyset$ implies $X \setminus Ux_i \notin I$, for any i.
Thus our supposition is wrong.
This proves that each ultrafilter on X is I−convergent.
Conversely, suppose each ultrafilter on X is I−convergent. We have to show that X is compact. Suppose the contrary that X is not compact. Then there is an open cover U' of X with no finite subcover.
Let $B = \{X \setminus \cup^n_{i=1} U_i : U_i \in U', i=1, 2, \ldots, n; n \in N\}$.
Then clearly, *B* is a non-empty family of non-empty subsets of X which is closed under finite intersection and so a filter base for some filter, say Ƒ on X. Since

every filter is contained in an ultrafilter, there is an ultrafilter Ģ on X such that Ƒ ⊂ Ģ. By the given condition, Ģ is I−convergent, where I = I(Ģ). Suppose $I - \lim$ Ģ $= x_0$. Then for each nbd U of x_0 , $\{V \in P(X) : U \cap V = \emptyset\} \subset I \cdots (**)$.
Now clearly, X \ U ∈ **B** and so X \ U ∈ Ģ.
Now U, X \ U ∈ Ģ implies U ∩ (X \ U) ∈ Ģ. That is, ∅ ∈ Ģ, which is not true. Thus our suppositon is wrong.
Hence X is compact.

We recall the following:
Maps between the sets can be put to act on ultrafilters. More precisely, one has the following construction.
Suppose f : X → Y is a map and Ų is an ultrafilter on X. Consider the collection
f^* (Ų) = {V ⊂ Y : f^{-1} (V) ∈ Ų }. Then f^* (Ų) is clearly an ultrafilter on Y. With the above notations, we have f (U) ∈ f^* (Ų), ∀ U ∈ Ų.

Proposition 2.8 Let $x_0 \in X$ and f : X → Y be a map. Then f is continuous at x_0 if and only if whenever Ų is an ultrafilter on X with $I_X - \lim$ Ų $= x_0$, then f^* (U) is an ultrafilter on Y with $I_Y - \lim f^*$(Ų) = f (x_0), where $I_X = I_X$ (Ų) and $I_Y = I_Y$ (f^* Ų).

Proof. First suppose f : X → Y is continuous at x_0. Let Ų be an ultrafilter on X such that $I_X - \lim$ Ų $= x_0$. Then for each nbd U of x_0, {W ∈ P(X) : U ∩ W = ∅}⊂ IX • • • (∗). By above recall f∗ (Ų) is an ultrafilter on Y. We have to show that IY − lim f (Ų) = f (x0). For this, let V be a nbd of f (x0) in Y. We claim that {T ∈ P(Y) : V ∩ T = ∅} ⊂ IY . So, let T ∈ P(Y) such that V ∩ T = ∅. Since f is continuous at x0, for above nbd V of f (x0), there exists a nbd U of x0 such that f (U) ⊂ V. Now, V ∩ T = ∅ implies that T ⊂ Y \ V ⊂ Y \ f (U) • • • (∗∗).
Now, from (∗), U ∩ (X \ U) = ∅ implies that X \ U ∈ IX and so U ∈ Ų. This further implies that f (U) ∈ f(Ų). Thus Y \ f (U) ∈ IY . Since ideal is closed under subsets, from (∗∗), T ∈ IY .
This proves that IY − lim f(Ų) = f (x0).
Conversely, suppose the condition holds. We have to show that f is continuous at x_0. Suppose the contrary that f is not continuous at x_0. Then there exists a nbd V of f (x_0) such that f^{-1} (V) is not a nbd of x_0. Consider Ƒ = {U \ f^{-1} (V) :U is a nbd of x_0}. Our assumption on V shows that all the sets in Ƒ are non-empty. Otherwise, f^{-1} (V) would contain some nbd of x_0 , which would force f^{-1} (V) itself to be a nbd of x_0. It is clear that Ƒ is a filter on X. Let Ų be an ultra-filter on X containing Ƒ. We claim that $I_X - \lim$ Ų $= x_0$, where $I_X = I_X$ (Ų). For this, let U be a nbd of x_0. We need to show that {W ∈ P(X) : U ∩ W = ∅} ⊂ I_X
So let W ∈ P(X) such that U ∩ W = ∅.
Now, U ∩ W = ∅ implies W ⊂ X \ U · · · (∗ ∗ ∗).
We first show that Ųx_0 ⊂ Ų. Suppose not. Then there is a nbd U of x_0 such that U ∈ Ų. Then clearly, X \ U ∈ Ų. Now, X \ U, U \ f^{-1} (V) ∈ Ų implies that ∅ = (X \ U) ∩ (U \ f^{-1} (V)) ∈ Ų, which is not possible. Thus Ųx_0 ⊂ Ų. That is, U ∈ Ų, ∀ U ∈ Ųx_0. So, X \ U ∈ I_X , ∀ U ∈ Ųx_0. Since I_X is an ideal of X, from (∗ ∗ ∗), W ∈ I_X . This proves that {W ∈ P(X) : U ∩ W = ∅} ⊂ I_X.
Hence $I_X - \lim$ Ų $= x_0$.
By the given condition, $I_Y - \lim f*$ (Ų) = f (x_0).
Since V is a nbd of f (x_0) and f^* (Ų) is I_Y −convergent to f (x_0), it follows that V ∈ f^* (Ų), which means that f^{-1} (V) ∈ Ų. But this leads to a contradiction since X \ f^{-1} (V) clearly belongs to Ƒ ⊂ Ų.
Therefore, f must be continuous at x_0.

Characterization of Closure
Proposition 2.9 Let E ⊂ X. Then x_0 ∈ E if and only if there is an ultrafilter Ģ on X such that E ∈ Ģ and $I - \lim$ Ģ $= x_0$, where I = I(Ģ).

Proof. First suppose x_0 ∈ E. Then each nbd of x_0 meets E. That is, U ∩ E = ∅, ∀ U ∈ Ųx_0, where Ųx_0 is the nbd system at x_0. Let Ƒ = {U ∩ E : U ∈ Ųx_0 } ∪ Ųx_0. Then Ƒ is clearly a filter on X containing E.
Let Ģ be an ultrafilter on X containing Ƒ. Then E ∈ Ģ and U ∈ Ģ, ∀ U ∈ Ųx_0. We shall show that $I - \lim$ Ģ $= x_0$, where I = I(Ģ). For this, let U be a nbd of x_0. We claim that {V ∈ P(X) : U ∩ V = ∅} ⊂ I.
So, let V ∈ P(X) such that U ∩ V = ∅. Now, U ∩ V = ∅ implies that U ⊂ X \ V. Also, U ∈ Ģ and Ģ is a filter on X implies that X \ V ∈ Ģ. Thus V ∈ I.
Therefore, $I - \lim$ Ģ $= x_0$.
Converse is obvious using Proposition 1.15.

Theorem 2.10 An ultra-filter Ų I_X −converges to x in $X = \Pi_{\alpha\in\Lambda} X_\alpha$ if and only if p_α(Ų) $I_{X\alpha}$ −converges to $p_\alpha(x)$, ∀ α, where $I_X = I_X$ (Ų) and $I_{X\alpha} = I_{X\alpha}$ (p_α (Ų)).
Proof. It follows by Theorem 1.21.

III. EQUIVALENCE OF I-CONVERGENCE OF ULTRAFILTERS AND CONVERGENCE OF NETS

We start with the following terms.
We know that every filter is contained in an ultrafilter. An ultrafilter which contains a derived filter is called a *derived ultrafilter*.
Let Ƒ be an indexed filter on X with index set D. Any net $\lambda : D \to X$ with $\lambda(d) \in Ƒ_d$ is called a *derived net* of Ƒ.
A net $\lambda : D \to X$ in X is said to be *convergent* to $x_0 \in X$ if for each nbd U of x_0, there is some $d \in D$ such that $c \geq d$ in D implies that $\lambda(c) \in U$. In other words, some tail $\Lambda_d = \{\lambda(c) : c \geq d \text{ in } D\} \subset U$.

Theorem 3.1 An ultrafilter Ģ on X I–converges to x_0 in X if and only if every derived net λ of Ģ converges to x_0 in X, where I = I(Ģ).

Proof. Since every ultra-filter Ģ on X itself is a filter on X, the proof follows by Theorem 1.16.

Lemma 3.2 An ultra-filter Ģ on X I–converges to x_0 in X if and only if Ģ converges to x_0 in X

Proof. It follows from above Theorem 3.1 and the fact that an ultra-filter Ģ on X converges to x_0 in X if and only if every derived net of Ģ converges to x_0 in X.

Theorem 3.3 A net $\lambda: D \to X$ converges to x_0 in X if and only if the derived ultra-filter I–converges to x_0 in X.

Proof Suppose a net $\lambda: D \to X$ converges to x_0 in X. Then by Theorem1.17, the derived filter, say Ƒ I(Ƒ)–converges to x_0 and so the derived ultrafilter, say Ģ, I(Ƒ)–converges to x_0. By Proposition 1.14, the derived ultra-filter Ģ I (Ģ) –converges to x_0.
Converse follows clearly by Theorem 1.17.

IV. EQUIVALENCE OF I-CONVERGENCE OF ULTRAFILTERS AND I-CONVERGENCE OF NETS

Theorem 4.1 An ultrafilter Ģ on X I–converges to x_0 in X if and only if every derived net λ of Ģ I–converges to x_0 in X, where I = I(Ģ).

Proof. Since every ultrafilter Ģ on X itself is a filter on X, the proof follows by Theorem 1.18.

Lemma 4.2 An ultrafilter Ģ on X converges to x_0 in X if and only if every derived net λ of Ģ I–converges to x_0 in X.

Proof. It follows from above Theorem 4.1 and Lemma 1.19.

Theorem 4.3 Let $\lambda : D \to X$ be a net in X and Ģ be a derived ultrafilter of λ.Then λ I–converges to x_0 in X if and only if Ģ I–converges to x_0 in X, where I = I(Ģ).

Proof. It follows clearly by Theorem 1.20.

REFERENCES

[1] V. Balaz , J. Cervenansky, P. Kostyrko, T. Salat, I–convergence and I–continuity of real functions, Faculty of Natural Sciences, Constantine the Philosoper University, Nitra, Acta Mathematica 5, 43-50, 2002.

[2] N. Bourbaki, General Topology, Part (I) (transl.), Addison- Wesley, Reading ,1966.

[3] K. Demirci, I-limit superior and limit inferior, Math. Commun. 6 (2001), 165-172.

[4] H. Fast, sur la convergence statistique, colloq. Math. 2 (1951), 241-244.

[5] H. Halberstem, K. F. Roth, Sequences, Springer, New York, 1993.

[6] D. S. Jamwal, R. Jamwal, S. Sharma, I–Convergence of filters, New Trends in Mathematical Sciences, 4, No. 4, (2016), 322-328.

[7] R. Jamwal, S. Sharma and D. S. Jamwal, Some more results on I–Convergence of filters, New Trends in Mathematical Sciences, 5, No. 1, (2017), 190-195.

[8] R. Jamwal, Renu, D. S. Jamwal, I–Cluster points of filters, New Trends in Mathematical Sciences, 5, No. 4, (2017), 195-202.

[9] R. Jamwal, Renu, D.S.Jamwal, More on I–Cluster points of filters, New Trends in Mathematical Sciences, 5, No. 1, (2017), 203-208.

[10] P. Kostyrko, T.Salat, W. Wilczynski, I–convergence, Real Analysis, Exch. 26 (2) (2000/2001), 669-685.

[11] P. Kostyrko, M. Macaj, T.Salat, M. Sleziak, I–convergence and extremal I–limit points, Math. Slovaca, 55 (4) (2005), 443-464.

[12] B. K. Lahiri, P. Das, Further results on I–limit superior and I–limit inferior, Math. Commun.,8 (2003), 151-156.

[13] B. K. Lahiri, P. Das, I and I^* –convergence in topological spaces, Math. Bohemica, 130 (2) (2005), 153-160.

[14] B. K. Lahiri, P. Das, I and I^*–convergence of nets, Real Analysis Exchange, 33 (2) (2007/2008), 431-442.

[15] M. Macaj, T.Salat, Statistical convergence of subsequences of a given sequence, Math. Bohemica, 126 (2001), 191-208.

[16] M. Mursaleen, A. Alotaibi, On I–convergence in random 2–normed spaces, Math. Slovaca, 61(6) (2011) 933–940.

[17] M. Mursaleen, S. A. Mohiuddine, On ideal convergence of double sequences in probabilistic normed spaces, Math. Reports, 12(62)(4) (2010) 359-371.

[18] M. Mursaleen, S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, Math. Slovaca, 62(1) (2012) 49-62.

[19] M. Mursaleen, S. A. Mohiuddine, O. H. H. Edely, On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces, Comput. Math. Appl., 59 (2010) 603-611.

[20] I. Niven, H. S. Zuckerman, An introduction to the theory of numbers, 4th Ed., John Wiley, New York, 1980.

[21] T.Salat, On statistically convergent sequences of real numbers, Mathematical Slovaca, 30 (1980), No. 2, 139-150.

[22] T.Salat, B. C. Tripathy, M. Ziman, On I–convergence field, Italian J. of Pure Appl. Math. 17 (2005), 45-54.

[23] A.Sahiner, M. Gurdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math., 11(5) (2007), 1477-1484.

[24] I. J. Schoenberg, The integrability of certain function and related summability methods, Am. Math. Mon. 66 (1959), 361-375.

[25] S. Willard, General Topology, Addison-Wesley Pub. Co. 1970.