Solving Fourier Integral Problem by Using Laplace Transformation

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Abstract- **A linear ordinary differential equation with constant co-efficient, variable co-efficient and simultaneous differential equations is easily solved by using the Laplace transformation. The Laplace transformation is applicable in so many fields like engineering, Physics, Mathematics etc. Laplace transformation is used in solving the time domain function by converting it into frequency domain. In this paper the different types of Fourier transformation problems have been solved by using the Laplace Transformation.**

Index Terms- **Laplace transformation, Fourier Transformation.**

Sub area: Laplace transformation

Broad area: Mathematics

I. INTRODUCTION

It has been noticed that Laplace transformation is helpful for scientists, researches and engineers in number of ways. It is a mathematical tool which is used in the solving of differential equations by converting it from one form into another form. Generally it is effective in solving linear differential equation either ordinary or partial. It reduces an ordinary differential equation into algebraic equation. Ordinary linear differential equation with constant coefficient and variable coefficient can be easily solved by Laplace transformation method without finding the general solution and the arbitrary constant. It is used in solving different types of problems in Physics, material sciences etc. This involves ordinary/integral differential equation with constant and variable coefficients. It is also used to convert the signal system into frequency domain for solving it on a simple and easy way. We can also apply to explain non- homogeneous differential equations without solving the corresponding homogeneous differential equations. It has wide applications in different fields of engineering and technology besides basis sciences and mathematics.

II. FORMULATION

Let F (t) is a well defined function of t for all t ≥ 0. The Laplace transformation of F (t), denoted by f (p) or L {F (t)}, is defined as

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt = f(p)$$

Provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not. Where p the parameter which may be real or complex number and L is is the Laplace transformation operator.

The Laplace transformation of F (t) i.e. $\int_0^\infty e^{-pt} F(t)dt$ exists for p>a, if

F (t) is continuous and $\lim_{n\to\infty}\{e^{-at} F(t)\}$ is finite. It should however, be keep in mind that above condition are sufficient and not necessary.

Laplace transformation of elementary function:

1. $L\{1\} = \frac{1}{p}\ ,\ p > 0$
2. $L\{t^n\} = n!/p^(n+1)\ ,\ where\ n =$ 0,1,2,3 … … ….
3. $L\{e^{at}\} = \frac{1}{p-a}\ ,\ p > a$
4. $L\{sinat\} = \frac{a}{p^2+a^2}\ ,\ p > 0$
5. $L\{sinhat\} = \frac{a}{p^2-a^2}\ ,\ p > |a|$
6. $L\{cosat\} = \frac{p}{p^2+a^2}\ ,\ p > 0$
7. $L\{coshat\} = \frac{p}{p^2-a^2}\ ,\ p > |a|$

Proof: By the definition of Laplace transformation, we know that

1. $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$ then

$$L\{1\} = \int_0^\infty e^{-pt}\, 1\, dt$$

$$= -\frac{1}{p}\left(e^{-\infty} - e^{-0}\right) = \frac{1}{p}(0-1)$$

$$= \frac{1}{p} = f(p)\ ,p > 1$$

2. $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$ then.

$$L\{\ coshat\ \} = \int_0^\infty e^{-pt}\, coshat\ dt$$

$=\int_0^\infty e^{-pt}\left(\frac{e^{at}+e^{-at}}{2}\right)dt$

$=\int_0^\infty \left(\frac{e^{-(p-a)t}+e^{-(p+a)t}}{2}\right)dt$

$= -\frac{1}{2(p-a)}\,(e^{-\infty}+e^{-0})+\frac{1}{2(p+a)}\,(e^{-\infty}+e^{-0})$

$=\frac{1}{2(p-a)}+\frac{1}{2(p+a)}$

$=\frac{1}{2}\cdot\frac{2p}{p^2-a^2}$

Therefore, $L\{coshat\}=\frac{p}{p^2-a^2}\,,\ p>|a|$

Here Laplace transformation is applied on few examples.

(1) Now the Fourier integral problem $y=-\frac{2}{\pi}\int_0^\infty\left(\frac{coszy}{z^2}\right)dz$ may be solved by using Laplace transformation, as follow:

We know that the cosine form of Fourier integral representation is

$f(y)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\,coszt\,coszy\,dt\,dz$

$y=\frac{2}{\pi}\int_0^\infty\int_0^\infty t\,coszt\,coszy\,dt\,dz$

$y=\frac{2}{\pi}\int_0^\infty coszy\left[\int_0^\infty t\,coszt\,dt\right]dz$

……………………… (1)

Now we take

$$\int_0^\infty t\,coszt\,dt$$

…………………………… (2)

Now we can solve the equation (2) by both methods

Method –I (General method)

Solution of equation (2) is the real part of $\int_0^\infty t\,e^{-izt}\,dt$ ……… (3)

Now solve this let $izt=u\ ,than\ izds=du\ \Rightarrow dp=\frac{du}{iz}$

Therefore, $\int_0^\infty t\,e^{-izt}\,dt\ \Rightarrow\ \int_0^\infty \frac{u}{iz}\ e^{-u}\frac{du}{iz}\ dt\ \Rightarrow -\frac{1}{z^2}\int_0^\infty u\,e^{-u}du$

$-\frac{1}{z^2}\int_0^\infty u^{2-1}\,e^{-u}du\ \Rightarrow -\frac{1}{z^2}\ \Gamma 2\ \Rightarrow -\frac{1}{z^2}.1\Rightarrow -\frac{1}{z^2}$

Now from (3), $\int_0^\infty t\,e^{-izt}\,dt=-\frac{1}{z^2}$

$$\int_0^\infty t\,(coszt-isinzt)\,dt=-\frac{1}{z^2}$$

Equating the real parts on both sides

$$\int_0^\infty t\,coszt\,dt=-\frac{1}{z^2}$$

Now from equation (1)

$$y=\frac{2}{\pi}\int_0^\infty coszy\left[\int_0^\infty t\,coszt\,dt\right]dz$$

$$y=-\frac{2}{\pi}\int_0^\infty\frac{1}{z^2}coszy\ \ dz$$

This is the required solution.

Method –II (By using Laplace transformation)

Now we will solve the equation (2) by the Laplace transformation

From equation (2),

$$\int_0^\infty t\,coszt\,dt$$

We write this $\int_0^\infty e^{-0t}t\,coszt\,dt$

Comparing with L {F (t)} $=\int_0^\infty e^{-pt}F(t)dt=f(p)$

$where\ p=0$

Therefore $\int_0^\infty e^{-0t}t\,coszt\,dt$ = L $\{t\,coszt\}$

$(-1)^1\frac{d}{dp}\{f(p)\}$

$=(-1)^1\frac{d}{dp}\left(\frac{p}{z^2+p^2}\right)$

$\left[\frac{p^2-z^2}{(p^2+z^2)^2}\right]$

$Putting\ p=0$

Therefore, $\int_0^\infty t\,coszt\,dt=\ -\frac{z^2}{z^4}=-\frac{1}{z^2}$

Hence from (1),

$$y=\frac{2}{\pi}\int_0^\infty coszy\left[\int_0^\infty t\,coszt\,dt\right]dz$$

$$\boxed{y=-\frac{2}{\pi}\int_0^\infty\frac{1}{z^2}coszy\ \ dz}$$

This is the required solution.

On comparing these methods it was observed that the method –II is easy to solve such type of problem.

(2) Now another Fourier integral problem $y^3=\frac{2}{\pi}\int_0^\infty\left(\frac{6\,coszy}{z^4}\right)dz$ may be solved by using Laplace transformation, as follow:

We know that the cosine form of Fourier Integral Representation

$f(y)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\,coszt\,coszy\,dt\,dz$

Putting $f(y)=y^3$

$y^3=\frac{2}{\pi}\int_0^\infty\int_0^\infty t^3\,coszt\,coszy\,dt\,dz$

$y^3=\frac{2}{\pi}\int_0^\infty coszy\left[\int_0^\infty t^3\,coszt\,dt\right]dz$

……………………… (1)

Now we take

$$\int_0^\infty t^3\,coszt\,dt$$

…………………………… (2)

Here equation (2) is a higher power of integral problem , so it can be solved easily by using only Laplace transformation as follow:

Comparing equation (2) with L {F (t)} $=\int_0^\infty e^{-pt}F(t)dt=f(p)$

$where\ p=0$

Therefore $\int_0^\infty e^{-0t}t^3\,coszt\,dt$ = L $\{t^3 coszt\}$

$= (-1)^3 \frac{d^3}{dp^3}\{f(p)\} \Rightarrow$

$-\frac{d^3}{dp^3}\left(\frac{p}{z^2+p^2}\right)$

$= -\frac{d^2}{dp^2}\left[\frac{z^2-p^2}{(p^2+z^2)^2}\right] \Rightarrow$

$-2\frac{d}{dp}\left[\frac{p^3-3pz^2}{(p^2+z^2)^3}\right]$

$$-2\frac{d}{dp}\left[\frac{3(p^4-z^4)-6p(p^3-3pz^2)}{(p^2+z^2)^4}\right]$$

$$Putting\ \ p = 0$$

Hence, $\int_0^\infty e^{-0t} t\, coszt\, dt$

$= -\frac{6z^4}{z^8} = \frac{6}{z^4}$

From equation (1), $y^3 = \frac{2}{\pi}\int_0^\infty coszy\left[\int_0^\infty t^3\, coszt\, dt\right] dz$

$$y^3 = \frac{2}{\pi}\int_0^\infty \frac{6}{z^4} coszy\ \ dz$$

This is the required solution.

(3) Now we apply Laplace transformation on another example of Fourier integral Problem of fourth degree like

$y^4 = \frac{48}{\pi}\int_0^\infty\left(\frac{sinzy}{z^5}\right) dz$

We know that the sine form of Fourier Integral Representation

$f(y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\, sinzt\, sinzy\, dt\, dz$

Putting $f(y) = y^4$

$y^4 = \frac{2}{\pi}\int_0^\infty\int_0^\infty t^4\, sinzt\, sinzy\, dt\, dz$

$y^3 = \frac{2}{\pi}\int_0^\infty sinzy\left[\int_0^\infty t^4\, sinzt\, dt\right] dz$

……………………… (1)

Now we take

$$\int_0^\infty t^4\, sinzt\, dt$$

…………………………… (2)

Here again equation (2) is a higher power of integral problem , so it can be solved easily by using only Laplace transformation as follow:

Comparing equation (2) with L {F (t)} $=\int_0^\infty e^{-pt} F(t)dt = f(p)$

$$where\ p = 0$$

Therefore $\int_0^\infty e^{-0t} t^4\, sinzt\, dt$ = L { $t^4 sinzt$ }

$= (-1)^4 \frac{d^4}{dp^4}\{f(p)\} \Rightarrow$

$-\frac{d^4}{dp^4}\left(\frac{z}{z^2+p^2}\right)$

$= -\frac{d^3}{dp^3}\left[\frac{2zp}{(p^2+z^2)^2}\right] \Rightarrow$

$-2z\frac{d^2}{dp^2}\left[\frac{(z^2-3p^2)}{(p^2+z^2)^3}\right]$

$$-24z\frac{d}{dp}\left[\frac{(z^3-pz^2)}{(p^2+z^2)^4}\right]$$

$$= -24z\left[\frac{(p^2+z^2)(3p^2-z^2)-8p(p^3-pz^2)}{(p^2+z^2)^4}\right]$$

$$Putting\ \ p = 0$$

$\int_0^\infty t^4\, sinzt\, dt = -24z\left[\frac{z^4}{z^{10}}\right] = \frac{24z^5}{z^{10}} = \frac{24}{z^5}$

From equation (1), $y^4 = \frac{2}{\pi}\int_0^\infty sinzy\left[\int_0^\infty t^3\, sinzt\, dt\right] dz$

$$y^4 = \frac{2}{\pi}\int_0^\infty \frac{24}{z^5}\ sinzy\ \ dz \Rightarrow \frac{48}{\pi}\int_0^\infty \frac{sinzy}{z^5}\ \ dz$$

Hence, $\boxed{y^4 = \frac{48}{\pi}\int_0^\infty \frac{sinzy}{z^5}\ \ dz}$

This is the required solution.

III. CONCLUSION

In this paper we have applied the Laplace transformation method in solving Fourier Integral Problems. Some examples of Fourier Integral Problems have been solved by using the Laplace transformation. It has been noticed that this technique is very much capable in finding solutions of Fourier Integral Problems .The primary use of Laplace transformation is converting a time domain functions into frequency domain function. Here, Laplace transformation of some elementary functions has been discussed in details and found very effective to simplify the various Fourier Integral Problems.

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