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An overview of some special functions

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Abstract- **The Laplace transformation is a mathematical tool which is used in the solving of differential equations by converting it from one form into another form. Regularly it is effective in solving linear differential equations either ordinary or partial. The Laplace transformation is used in solving the time domain function by converting it into frequency domain function. Laplace transformation makes it easier to solve the problem in engineering application and make differential equations simple to solve. In this paper we will discuss the Laplace Transformation of some special special functions like, Sine Integral function, Cosine Integral function, Exponential Integral function, Error and Complementary Error function, Heaviside's Unit Function and Dirac delta function, Laguerre Polynomial.**

Index Terms- **Laplace transformation,**
***Sub area*: Laplace transformation**
***Broad area*: Mathematics**

INTRODUCTION

The Laplace transformation is applied in different areas of science, engineering and technology. The Laplace transformation is applicable in so many fields. It is effective in solving linear differential equation either ordinary or partial. It reduces an ordinary differential equation into algebraic equation. Ordinary linear differential equation with constant coefficient and variable coefficient can be easily solved by the Laplace transformation method without finding the generally solution and the arbitrary constant. It is used in solving physical problems. This involving integral and ordinary differential equation with constant and variable coefficient It has some applications in nearly all engineering disciplines, like System Modeling, Analysis of Electrical Circuit, Digital Signal Processing, Nuclear Physics, Process Controls, Applications in Probability, Applications in Physics, Applications in Power Systems Load Frequency Control, Mat lab etc.

DEFINITION

Let F (t) is a well defined function of t for all t ≥ 0. The Laplace transformation of F (t), denoted by f (p) or L {F (t)}, is defined as

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt = f(p)$$

Provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not. Where p the parameter which may be real or complex number and L is is the Laplace transformation operator.

The Laplace transformation of F (t) i.e. $\int_0^\infty e^{-pt} F(t)dt$ exists for p>a, if F (t) is continuous and $\lim_{n\to\infty}\{e^{-at}F(t)\}$ is finite. It should however, be keep in mind that above condition are sufficient and not necessary.

Formulation:

a. Sine Integral function.
b. Cosine Integral function.
c. Exponential Integral function.
d. Error and Complementary Error function.
e. Heaviside's Unit Function and Dirac delta function.
f. Laguerre Polynomial.

(A) Sine integral function:

The Sine integral function is denoted by

$$S_i(t) = \int_0^t \frac{sint}{t} dt$$

Therefore,

$$L\{S_i(t)\} = L\left[\int_0^t \frac{sint}{t} dt\right] = \frac{1}{p} L\left[\frac{sint}{t}\right] \ldots\ldots\ldots \quad (1)$$

But we know that, the property of division by t,

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(p)dp$$

Now,

$$L\left[\frac{Sint}{t}\right] = \int_s^\infty \frac{p}{p^2+1} dp = \frac{\pi}{2} - \tan^{-1} p = \cot^{-1} p$$

Hence from (1),

$L\{S_i(t)\} = L\left[\int_0^t \frac{sint}{t}dt\right] = \frac{1}{p}\cot^{-1}p$

Hence, $L\{S_i(t)\} = \frac{1}{p}\cot^{-1}p$

(B) Cosine Integral function:

The cosine integral function is denoted by

$$C_i(t) = \int_0^t \frac{cost}{t}dt$$

Therefore,

$$L\{C_i(t)\} = L\left[\int_0^t \frac{cost}{t}dt\right] \text{……..} \quad (1)$$

Let

$$F(t) = \int_t^\infty \frac{cost}{t}dt \Rightarrow F'(t) = -\frac{cost}{t}$$

$$\Rightarrow tF'(t) = -cost$$

Therefore, $L\{tF'(t)\} = -L\{cost\}$

$$\frac{d}{dp}\{pf(p) - F(0)\} = -\frac{p}{p^2+1}$$

$$\frac{d}{dp}\{pf(p)\} = \frac{p}{p^2+1}$$

$$because\ F(0) is\ constant$$

Integrating both sides with respect to p, we get,

$$pf(p) = \frac{1}{2}\log(p^2+1) + k$$

or $\lim_{p\to 0}\{pf(p)\} = \lim_{p\to 0}[\frac{1}{2}\log(p^2+1) + k]$

$$= 0 + k = k$$

or $\lim_{t\to\infty}\{F(t)\} = k \quad or \quad 0 = k$

Hence,

$$pf(p) = \frac{1}{2}\log(p^2+1)$$

or

$$f(p) = \frac{\log(p^2+1)}{2p}$$

Hence, $L\{F(t)\} = f(p) = \frac{\log(p^2+1)}{2p}$

Hence from (1),

$$L\{C_i(t)\} = L\{F(t)\} = \frac{\log(p^2+1)}{2p}$$

(C) Exponential Integral function:

The exponential integral function is defined by

$$E_i(t) = \int_t^\infty \frac{e^{-y}}{y}dy$$

Let $F(t) = \int_t^\infty \frac{e^{-y}}{y}dy$ ………. (1)

Therefore,

$$L\{tF'(t)\} = -L\{e^{-t}\} = -\frac{1}{p+1}$$

$$\frac{d}{dp}\{pf(p) - F(0)\} = -\frac{1}{p+1}$$

$$\frac{d}{dp}\{pf(p)\} = \frac{1}{p+1}$$

$$because\ F(0) is\ constant$$

Integrating both sides with respect to p, we get,

$$pf(p) = \log(p+1) + k$$

Now,

$$\lim_{p\to 0}\{pf(p)\} = \lim_{p\to 0}[\log(p+1) + k]$$

$$= 0 + k = k$$

Hence,

$$\lim_{t\to\infty}\{F(t)\} = k \quad or \quad 0 = k$$

Hence,

$$pf(p) = \log(p+1)$$

or

$$f(p) = \frac{\log(p+1)}{p}$$

Hence, $L\{F(t)\} = f(p) = \frac{\log(p+1)}{p}$

Hence from (1),

$$L\{E_i(t)\} = L\{F(t)\} = \frac{\log(p+1)}{p}$$

(D) Error and Complementary Error function:

The error function is defined by $\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-y^2}dy$

$$\text{L}\{\text{erf}(\sqrt{t})\} = L\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-y^2}dy\right\} \text{….}(1)$$

$$\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}(1 - y^2 + \frac{y^4}{2!} - \frac{y^6}{3!} + \cdots \text{….})dy$$

$$\frac{2}{\sqrt{\pi}}\left[y - \frac{y^3}{3} + \frac{y^5}{5.2!} - \frac{y^7}{7.3!} + \cdots\right]$$

$$\frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots\right]$$

Hence,

$$\text{L}\{\text{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}}L\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{p^{3/2}} - \frac{1}{3}\frac{\Gamma 5/2}{p^{5/2}} + \frac{1}{5.2!}\frac{\Gamma 7/2}{p^{7/2}} \text{….}\right]$$

$$\frac{1}{p^{3/2}}\left[1 - \frac{1}{2}\frac{1}{p} + \frac{1}{2}.\frac{3}{4}.\frac{1}{p^2} - \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{1}{p^3}\text{…}\right]$$

$$\frac{1}{p^{3/2}}\left[1 + \frac{1}{p}\right]^{-1/2} = \frac{1}{p^{3/2}}\left[\frac{p}{p+1}\right]^{1/2}$$

$$\frac{1}{p\sqrt{p+1}}$$

Then from (1),

$$\mathrm{L}\{\mathrm{erf}(\sqrt{t})\} = \frac{1}{p\sqrt{p+1}}$$

Complementary error function:

This function is defined by

$$erf_c\sqrt{t} = 1 - \mathrm{erf}(\sqrt{t})$$

Taking Laplace transformation on both sides

$L\{erf_c\sqrt{t}\} = L\{1 - \mathrm{erf}(\sqrt{t})\}$

$L\{erf_c\sqrt{t}\} = L\{1\} - \mathrm{L}\{\mathrm{erf}(\sqrt{t})\}$

$$L\{erf_c\sqrt{t}\} = \frac{1}{P} - \frac{1}{p\sqrt{p+1}}$$

(E) Heaviside's Unit Function and Dirac delta function:

Heaviside's Unit Function:

This function is defined by

$$F(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & t > a \end{cases}$$

We know that by definition of Laplace transformation

$$\mathrm{L}\ \{\mathrm{F}\ (\mathrm{t})\} = \int_0^\infty e^{-pt} F(t)\, dt$$

Therefore, $L\{F(t-a)\} = \int_0^\infty e^{-pt} F(t-a) dt$

$$= \int_0^a e^{-pt} F(t-a) dt + \int_a^\infty e^{-pt} F(t-a) dt$$

$= \int_0^a e^{-pt}.0\, dt + \int_a^\infty e^{-pt}.1\, dt$

$$L\{F(t-a)\} = 0 + \frac{e^{-pa}}{p} = \frac{e^{-pa}}{p}$$

Hence,

$$L\{F(t-a)\} = \frac{e^{-pa}}{p}$$

Dirac delta function:

This function is defined by

$$(F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$$

We know that by definition of Laplace transformation

$$\mathrm{L}\ \{\mathrm{F}\ (\mathrm{t})\} = \int_0^\infty e^{-pt} F(t)\, dt$$

Therefore

$\mathrm{L}\ \{F_\epsilon(t)\} = \int_0^\infty e^{-pt} F_\epsilon(t) dt$

$$= \int_0^\epsilon e^{-pt} F_\epsilon(t) dt + \int_\epsilon^\infty e^{-pt} F_\epsilon(t) dt$$

$$\frac{1}{p\epsilon}(e^0 - e^{p\epsilon}) = \frac{1}{p\epsilon}(1 - e^{p\epsilon})$$

$$Hence, \quad \mathrm{L}\ \{F_\epsilon(t)\} = \frac{1}{p\epsilon}(1 - e^{p\epsilon})$$

(F)Laguerre Polynomial

The Laguerre polynomial is defined as

$$L_n(u) = \frac{e^u}{n!}\frac{d^n}{du^n}(e^{-u}u^n)$$

We know that by the definition of Laplace Transform

$$\mathrm{L}\ \{\mathrm{F}\ (\mathrm{t})\} = \int_0^\infty e^{-pt} F(t)\, dt$$

Therefore,

$\mathrm{L}\ \{L_n(t)\} = \int_0^\infty e^{-pt}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)\right\} dt$

$$= \frac{1}{n!}\int_0^\infty e^{-(p-1)t}\left\{\frac{d^n}{dt^n}(e^{-t}t^n)\right\} dt$$

$= \frac{1}{n!}[(p-1)\int_0^\infty e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt]$

Integrating again,

$$\frac{(p-1)^2}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^{n-2}}{dt^{n-2}}(e^{-t}t^n)dt$$

Integrating n again,

$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^{n-n}}{dt^{n-n}}(e^{-t}t^n)dt$$

$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}(e^{-t}t^n)dt = \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt}t^n dt$

But by the definition of Laplace Transformation

$$\mathrm{L}\ \{\mathrm{F}\ (\mathrm{t})\} = \int_0^\infty e^{-pt} F(t)\, dt$$

Hence,

$$\frac{(p-1)^n}{n!}L(t^n)$$

$$= \frac{(p-1)^n}{n!}.\frac{n!}{p^{n+1}}$$

Hence,

$$\mathrm{L}\ \{L_n(t)\} = \frac{(p-1)^n}{p^{n+1}}$$

CONCLUSION

This paper presents a brief overview of Laplace transformation of some special functions. The primary use of Laplace transformation is converting a time domain functions into frequency domain function. The Laplace transformation of some special function like Sine Integral function, Cosine Integral function, Exponential Integral function Error and Complementary Error function, Heaviside's Unit Function and Dirac delta function, Laguerre Polynomial were discussed in detail. Laplace transformation is a very useful mathematical tool to make simpler complex problems in the area of stability and control.

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