Numerical Solution of Fuzzy Differential Equation

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***Abstract* - In this dissertation, a method of estimating the approximate solution of fuzzy differential equation with initial conditions in order to increase the exactness of the solution. The euler method and runge kutta method is discussed in detail. Finally, a comparison and complete error of both the method is discussed.**

***Index Terms* - Fuzzy sets, Fuzzy Differential equations, Fuzzy Cauchy problem, Runge kutta method.**

1.INTRODUCTION

Fuzzy differential equations have been growing in recent years. Chang and Zadeh [5] were first introduced the concept of fuzzy derivatives. It was followed up by Dubios and Prade [6] and used the extension principles. Differential of fuzzy functions was contributed by Puri and Ralesec[11] and Goetschel and Voxman[7]. Kaleva [8,9] and Seikkala[12] widely studied the fuzzy differential equation and initial value problems.

Numerical solution of fuzzy differential equations has been introduced by Ma, Friedman, Kandel [10] through Euler method and by Abbasbandy and Allahviranloo[1] by Runge-Kutta method. This paper is organized as follows:

In section 2, some results on fuzzy Cauchy problem, fuzzy derivatives. In section 3, we propose the Runge Kutta method to solve the fuzzy differential equations. The numerical example is given in section 4. The comparison of both the methods are implemented with its approximation solution and its complete error is found.

2. PRELIMINARIES

2.1 Fuzzy sets

The idea of fuzzy set was introduced by Lotfi Zadeh in 1965 as a means of handling uncertainty that is due to imprecision or vagueness rather than to randomness. Fuzzy sets were taken up with interests by engineers, computer scientists and operations researchers. While mathematicians have been involved with the development of fuzzy sets from the very beginning, it has really been in recent years only that fuzzy sets have started receiving serious consideration from a wider mathematical community. Many interesting mathematical problems are coming and the mathematical foundations of the subject are firmly established and now it has emerged as an independent branch of applied Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade u(x) taking values in [0,1], with u(x) = 0 corresponding to non-membership, 0 < u(x) < 1 to partial membership, and = 1 to full membership. According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u: X \to [0,1]$. The function u itself used for the fuzzy set.

2.2 Fuzzy cauchy problem

Consider the fuzzy initial value problem equation y' = f (t, y)

$$y'(t) = f(t, y(t)), 0 \le t \le T, \quad y(t_0) = y_0,$$

Where f is a conituous mapping from

$$\Re_+ \times \Re \to \Re and\ y_0 \in E$$

with r-level sets

$$\lfloor y_0 \rfloor_r = \lfloor \underline{y}(0:r), \overline{y}(0:r) \rfloor, r \in [0,1].$$

The extension principle of Zadeh leads to the following definition of f(t,y), where y=y(t) is a fuzzy number.

$$f(t,y)(s) = \sup\{y(\tau) \setminus s = f(t,\tau)\}, s \in \Re$$

$$\Rightarrow [f(t,y)]_r = \lfloor \underline{f}(t,y:r), \overline{f}(t,y:r) \rfloor, r \in [0,1]$$

It follows that

$$\underline{f}(t,y:r) = \min \{f(t,u) \setminus u \in \lfloor \underline{y}(r), \overline{y}(r) \rfloor\} \text{ and}$$

$$\overline{f}(t,y:r) = \max \{\{f(t,u) \setminus u \in \lfloor \underline{y}(r), \overline{y}(r) \rfloor\}.$$

2.3 Interpolation of fuzzy number

The problem of interpolation for fuzzy sets is as follows:

Suppose that at various time instant t information f(t) is presented as fuzzy set. The aim is to approximate the function f(t) , for all t in the domain of f.

Let $t_0 < t_1 < \dots < t_n$ be n+1 distinct points in R and let $u_0, u_1, \dots\dots, u_n$ be n+1 fuzzy sets in E.

A fuzzy polynomial interpolation of the data is a fuzzy value continuous function f: R → E satisfying:

(i) $f(t_i) = \tilde{u}_i$, i= 1,…..,n.

(ii) If the data is crisp, then the interpolation f is a crisp polynomial.

A function f which fulfilling these condition may be constructed as follows.

Let $C^i_\infty = [u_i]$ for any $\alpha \in [0,1],$ i=0,1,2,….,n.

For each $x = (x_0, x_1, \dots, x_n) \in R^{n+1}$, the unique polynomial of degree ≤ n denoted by P_X such that

$$P_X(t_i) = x_i, i = 0,1,2,\dots,n,$$

$$P_X(t) = \sum_{i=0}^{n} x_i \left(\prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \right).$$

Finally, for each $t \in R$ and all $\xi \in R$ is defined by $f(t) \in E$ by

$(f(t))(\xi) = \sup\{\alpha \in [0,1] : \exists X \in C^0_\alpha \times \dots\dots\times C^n_\alpha$ such that $P_X(t) = \xi$ }.

The interpolation polynolmial can be written level set wise as

$[f(t)]^\alpha = \{y \in R : Y = P_X(t), x \in [u_i]^\alpha, i = 1,2,\dots.,n\},$

for $0 \le \alpha \le 1$.

When the data $\tilde{u}_i$ presents as triangular fuzzy numbers, values of the interpolation polynomial are triangular fuzzy numbers. Then f(t) has a particular simple form that is well situated to computation.

3. SIXTH ORDER RUNGE-KUTTA METHOD

Let the exact solution of the given equation $[y(t)]_r = \lfloor \underline{y}(t:r), \overline{y}(t:r) \rfloor$ is approximate by some $[y(t)]_r = \lfloor \underline{y}(t:r), \underline{y}(t:r) \rfloor$ and we define

$\underline{y}(t_{n+1}:r) - \underline{y}(t_n:r) = \sum_{i=1}^{l} w_i \underline{k}_i$ and $\overline{y}(t_n:r) - \underline{y}(t_n:r) = \sum_{i=1}^{l} w_i \overline{k}_i$

where w_i's are constants,

$[k_i(t, y(t,r))]_\gamma = \lfloor \underline{k}_i(t, y(t,r)), \overline{k}_i(t, y(t,r)) \rfloor$ where i=1,2,3,4,5,6,7

$$\underline{k}_1(t, y(t:r)) = hf(t_n, \underline{y}(t_n:r))$$

$$\overline{k}_1(t, y(t:r)) = hf(t_n, \overline{y}(t_n:r))$$

$$\underline{k}_2(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n:r) + v\underline{k}_1\right)$$

$$\overline{k}_2(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \overline{y}(t_n:r) + v\overline{k}_1\right)$$

$$\underline{k}_3(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n:r) + \left((4v-1)\underline{k}_1 + \underline{k}_2\right)/8v\right)$$

$$\overline{k}_3(t,y(t:r)) = hf\left(t_n+\frac{h}{2},\overline{y}(t_n:r)+\left((4v-1)\overline{k}_1+\overline{k}_2\right)/8v\right)$$

$$\underline{k}_4(t,y(t:r)) = hf\left(t_n+\frac{2h}{3},\underline{y}(t_n:r)+\left((10v-2)\underline{k}_1+2\underline{k}_2+8v\underline{k}_3\right)/27v\right)$$

$$\overline{k}_4(t,y(t:r)) = hf\left(t_n+\frac{2h}{3},\overline{y}(t_n:r)+\left((10v-2)\overline{k}_1+2\overline{k}_2+8v\overline{k}_3\right)/27v\right)$$

$$\underline{k}_5(t,y(t:r)) = hf\left(t_n+(7+4.582576)\frac{h}{14},\underline{y}(t_n:r)+\begin{pmatrix}-((77v-56)+(17v-8)4.582576)\underline{k}_1\\ -8(7+4.582576)\underline{k}_2+48(7+4.582576)v\underline{k}_3\\ -3(21+4.582576)v\underline{k}_4\end{pmatrix}/392v\right)$$

$$\overline{k}_5(t,y(t:r)) = hf\left(t_n+(7+4.582576)\frac{h}{14},\overline{y}(t_n:r)+\begin{pmatrix}-((77v-56)+(17v-8)4.582576)\overline{k}_1\\ -8(7+4.582576)\overline{k}_2+48(7+4.582576)v\overline{k}_3\\ -3(21+4.582576)v\overline{k}_4\end{pmatrix}/392v\right)$$

$$\underline{k}_6(t,y(t:r)) = hf\left(t_n+(7-4.582576)\frac{h}{14},\underline{y}(t_n:r)+\begin{pmatrix}-5((287v-56)-(59v-8)4.582576)\underline{k}_1\\ -40(7+4.582576)\underline{k}_2+320(4.582576)v\underline{k}_3\\ +3(21-121(4.582576)v\overline{k}_4+392(6-(4.582576)v\overline{k}_5\end{pmatrix}/1960v\right)$$

$$\overline{k}_6(t,y(t:r)) = hf\left(t_n+(7-4.582576)\frac{h}{14}\overline{y}(t_n:r)+\begin{pmatrix}-5((287v-56)-(59v-8)4.582576)\overline{k}_1\\ -40(7-4.582576)\overline{k}_2+320(4.582576)v\overline{k}_3\\ +3(21-121(4.582576)v\overline{k}_4+392(6-(4.582576)v\overline{k}_5\end{pmatrix}/1960v\right)$$

$$\underline{k}_7(t,y(t:r)) = hf\left(t_n+h,y(t_{n:}:r)+\begin{pmatrix}15((30v-8)7v(4.582576))\underline{k}_1+120\underline{k}_2-40(5+7(4.582576))v\underline{k}_3\\ +63(2+3(4.582576))v\underline{k}_4-14(49-9(4.582576))v\underline{k}_5\\ +70(7+(4.582576))v\underline{k}_6\end{pmatrix}/180v\right)$$

$$\overline{k}_7(t,y(t:r)) = hf\left(t_n+h,y(t_{n:}:r)+\begin{pmatrix}15((30v-8)7v(4.582576))\overline{k}_1+120\overline{k}_2-40(5+7(4.582576))v\overline{k}_3\\ +63(2+3(4.582576))v\overline{k}_4-14(49-9(4.582576))v\overline{k}_5\\ +70(7+(4.582576))v\overline{k}_6\end{pmatrix}/180v\right)$$

$$F(t,y(t:r)) = 9\underline{k}_1(t,y(t:r))+64\underline{k}_3(t,y(t:r))+49\underline{k}_5(t,y(t:r))+49\underline{k}_6(t,y(t:r))+9\underline{k}_7(t,y(t:r))$$

$$G(t,y(t:r)) = 9\overline{k}_1(t,y(t:r))+64\overline{k}_3(t,y(t:r))+49\overline{k}_5(t,y(t:r))+49\overline{k}_6(t,y(t:r))+9\overline{k}_7(t,y(t:r))$$

The exact and approximate solution at $t_n, 0 \le n \le N$ are denoted by

$$[Y(t_n)]_r = \lfloor \underline{Y}(t_n:r),\overline{Y}(t_n:r)\rfloor \; and \; [y(t_n)]_r = \lfloor \underline{y}(t_n:r),\overline{y}(t_n:r)\rfloor$$

The solution calculated by grid points at a= $t_0 \le t_1 \le t_2 \le \ldots \le t_N = b$ and $h = \dfrac{b-a}{n} = t_{n+1} - t_n$. Therefore we have,

$$\underline{Y}(t_{n+1} : r) = \underline{Y}(t_n : r) + \frac{1}{180} F\left[t_n, \underline{Y}(t_n : r)\right]$$

$$\overline{Y}(t_{n+1} : r) = \overline{Y}(t_n : r) + \frac{1}{180} G\left[t_n, \overline{Y}(t_n : r)\right]$$

$$\underline{y}(t_{n+1} : r) = \underline{y}(t_n : r) + \frac{1}{180} F\left[t_n, \underline{y}(t_n : r)\right]$$

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$$\overline{y}(t_{n+1} : r) = \overline{y}(t_n : r) + \frac{1}{180} G\left[t_n, \overline{y}(t_n : r)\right]$$

Here, we show the convergence of these approximations as

$$\lim_{k \to 0} \underline{y}(t : r) = \underline{Y}(t : r) and \lim_{k \to 0} \overline{y}(t : r) = \overline{Y}(t : r)$$.

4. NUMERICAL EXAMPLE

R	6th ORDER RUNGE- KUTTA METHOD		EXACT SOLUTION	
0.1	2.1066684723	3.2075719833	2.1066684171	3.2075725576
0.2	2.1746253967	3.1532073021	2.1746254628	3.1532069210
0.3	2.2425825596	3.0988414288	2.2425825085	3.0988412844
0.4	2.3105394840	3.0444755554	2.3105395542	3.0444756479
0.5	2.3784964085	2.9901101589	2.3784967999	2.9901100113
0.6	2.4464540482	2.9357445240	2.4464536456	2.9357443747
0.7	2.5144107342	2.8813788891	2.5144106913	2.8813787382
0.8	2.5823678970	2.8270127773	2.5823677370	2.8270131016
0.9	2.6503250599	2.7726476192	2.6503247827	2.7726474650
1.0	2.7182817459	2.7182817459	2.1782818285	2.7182818285

Example 4.1
Consider the fuzzy initial value problem,

$$y'(t) = y(t), t \in [0,1] with$$
$$y(0) = (0.75 + 0.25r, 1.2 - 0.2r)$$
$$where 0 \le r \le 1$$

Solution:
The exact solution is given by

$$\underline{y}(t : r) = \underline{y}(t : r) e^{t} and \overline{y}(t : r) = \overline{y}(t : r) e^{t}$$

Then at t=1,
Table 4.1
y(1: r) =

$$\left[(0.75 + 0.25r)e, (1.2 - 0.2r)e\right], 0 \le r \le 1.$$

By using Runge-Kutta sixth order method the following results are obtained:

Example 4.2
Consider the fuzzy initial value problem,

$$y'(t) = y(t), t \in I = [0,1],$$
$$y(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], 0 < \alpha \le 1$$
$$y(0.1) = [(0.75 + 0.25\alpha) e^{0.1}, (1.125 - 0.125\alpha) e^{0.1}],$$
$$y(0.2) = [(0.75 + 0.25\alpha) e^{0.2}, (1.125 - 0.125\alpha) e^{0.2}],$$
$$y(0.3) = [(0.75 + 0.25\alpha) e^{0.3}, (1.125 - 0.125\alpha) e^{0.3}],$$
$$y(0.4) = [(0.75 + 0.25\alpha) e^{0.4}, (1.125 - 0.125\alpha) e^{0.4}],$$

The exact solution at t=1 is given by $Y(1;\alpha) = [(0.75+0.25\alpha)e, (1.125-0.125\alpha)e], 0<\alpha\le 1.$

By using Adam's-fifth order predictor corrector method the following results are obtained:

Table 4.2

r	ADAM'S ORDER 5		EXACT SOLUTION	
0.1	2.106668505	3.024088534	2.106668417	3.024088534
0.2	2.174625553	2.990110011	2.174625462	2.990110011
0.3	2.242582602	2.956131488	2.242582508	2.956131488
0.4	2.310539650	2.922152966	2.310539554	2.922152965
0.5	2.378496699	2.888174443	2.378496601	2.888174443
0.6	2.446453748	2.854195919	2.446453645	2.854195920
0.7	2.514410796	2.820217397	2.514410691	2.820217398
0.8	2.582367845	2.786238990	2.582367737	2.786238874
0.9	2.650324893	2.752260466	2.605324782	2.752260351
1.0	2.718281942	2.718281942	2.718281828	2.718281828

The comparison are obtained by Adam's method fifth order and runge kutta sixth order method is given in the table 4.3.

Table 4.3

r	ADAM'S FIFTH ORDER		RUNGE KUTTA SIXTH ORDER	
0.1	8.7741×10^{-8}	1.2595×10^{-7}	5.52×10^{-8}	5.743×10^{-7}
0.2	9.0571×10^{-8}	1.2453×10^{-7}	6.61×10^{-8}	3.811×10^{-7}
0.3	9.3401×10^{-8}	1.2312×10^{-7}	5.11×10^{-8}	1.444×10^{-7}
0.4	9.6231×10^{-8}	1.2171×10^{-7}	7.02×10^{-8}	9.25×10^{-8}
0.5	9.9062×10^{-8}	1.2028×10^{-7}	1.914×10^{-7}	1.476×10^{-7}
0.6	1.0189×10^{-7}	1.1887×10^{-7}	4.026×10^{-7}	1.493×10^{-7}
0.7	1.0472×10^{-7}	1.1745×10^{-7}	4.29×10^{-8}	1.509×10^{-7}
0.8	1.0755×10^{-7}	1.1604×10^{-7}	1.600×10^{-7}	3.243×10^{-7}
0.9	1.1038×10^{-7}	1.1462×10^{-7}	2.772×10^{-7}	1.542×10^{-7}
1.0	1.1321×10^{-7}	1.1321×10^{-7}	8.26×10^{-8}	8.26×10^{-8}

5.CONCLUSION

In this paper, we have applied iterative solution of adam's predictor corrector fifth order method and runge kutta sixth order method for finding the numerical solution of fuzzy differential equations. Comparison of solution of example 1 and 2 shows that our proposed method gives better solution than Adam's fifth order predictor corrector method. We have proved that the sixth order Runge-Kutta Method proposed by us gives better solution than the adam's fifth order method.

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