A Study on Numerical Method of Differential Equation with Fuzzy Values

Ms. D. KalaiSelvi
Assistant Professor, Department of Mathematics, KG College of Arts and Science, Coimbatore

***Abstract* - In this paper, we study differential equation with fuzzy value. We propose a numerical method to approximate the fuzzy solution by using partition of fuzzy interval and generalization of Hukuhara difference and division. We prove some theorems for differential equation by fuzzy value.**

***Index Terms* - Fuzzy convex, Hukuhara difference, Monotonic, Matric space.**

I.INTRODUCTION

Fuzzy set theory is used to study a variety of problems fuzzy metric spaces [15], fuzzy linear systems [4, 5, 19], fuzzy differential equations [6, 7, 10, 16, 17] and other topics. The concept of fuzzy numbers and arithmetic operations with this numbers were first introduce and investigated by Chang and Zadeh [9] and others. Chang and Zadeh [9] first introduced the concept of the fuzzy derivative and followed by Dubois and Prade [10]. The concept of differential equations in a fuzzy environment was first formulated by Kaleva [12]. Several authors have produced a wide range of results in both the theoretical and applied fields of fuzzy differential equations [1, 2, 8, 11, 14, 17, 18]. Some of researchers worked for approximate solving the fuzzy initial value problem $y' = f(x, y)$ where x_0 is real number and $y(x_0) = y_0$ fuzzy number [1, 2, 8]. We consider the different fuzzy initial value problem $y = f(x, y)$ where x_0 and $y(x_0) = y_0$ are fuzzy numbers. We used of definition fuzzy directed line induced by L. Hongliang et al. [13] and extent to fuzzy interval. This paper used of partition of fuzzy interval [13] and generalization of Hukuhara difference and division [20]. we provide some background on fuzzy numbers and fuzzy differential equations. we present numerical method of fuzzy differential equation with full fuzzy initial values and give a numerical example for illustrate this method. Finally, conclusion is present.

II PRELIMINARIES

First, we review fuzzy numbers and some results about it. There are various definitions for the concept of fuzzy number. Let $E1$ be the set of all functions.
$u: R \to [0, 1]$ such that u is normal, fuzzy convex, upper semicontinuous and the closure of $\{x \in R: u(x) > 0\}$, is compact. For any a $\in E$, u is called a fuzzy number in parametric form a pair $(\underline{a}(r), \overline{a}(r))$ of function $\underline{a}(r)$, $\overline{a}(r)$, $0 \le r \le 1$ which satisfies the following requirements:

1. $\underline{a}(r)$ is a bounded monotonic increasing left continuous function.
2. $\overline{a}(r)$ is a bounded monotonic decreasing left continuous function.
3. $\underline{a}(r) \le \overline{a}(r)$,$0 \le r \ge 1$.

In this paper, we used of parametric form of fuzzy numbers. For a, b $\in E1$, the metric distance is defined as

D (u, v) = $\sup_{r\in[0,1]} \max\{ |\underline{a}(r) - \underline{b}(r)| , |\overline{a}(r) - \overline{b}(r)|$ (1.1)

Theorem2.1.[2],

1. (E, D) is a complete metric space;
2. D (a + c, b +c) = D(a,b) here a,b,c ∈ E;
3. D (a +b, c + e) ≤ D(a,c) + D(b, e), here a, b, c, e ∈ E.

For ranking of a, $b \in E$, $a \le b$ if and only if $\underline{a}(r) \le \underline{b}(r)$ and $\overline{a}(r) \le \overline{b}(r)$ and a < b if and only if $\overline{a}(r) < \overline{b}(r)$ for any r ϵ [0,1]

Definition 2.1. [13], Let $a_0, b_0 \in E$, $\overline{a}_0(r) < \overline{b}_0(r)$. The fuzzy number set
$\{c_t \in E | c_t = (1-t) a_0 + t b_0, t \in (-\infty, +\infty)\}$ is called fuzzy directed line induced by a_0, b_0 and denoted by $\overrightarrow{a_0 b_0}$.

Theorem 2.2.[13], Let ws, wt ∈ $\overrightarrow{a_0 b_0}$. then
(1) s ≤ t ⟺ ws ≼ wt
(2) s = t ⟺ ws = wt,i.e. s ≠t ⟺ ws ≠ wt .

Theorem 2.3. ([20]: Proposition 6), A⊖g B exists if and only if B ⊖g A and (−B) ⊖g (−A) exist and A ⊖g B = (−B) ⊖g (−A) = −(B ⊖g A).

Definition 2.3. [13], Let F: $\overrightarrow{a_0 b_0}$ → E be a fuzzy mapping and x ∈ R fixed.
Suppose
for |y| > |x|, xy > 0, F (wy) ⊖ F (wx) exists, and when x > 0

$$\frac{F(w_y) \ominus F(w_z)}{w_y \ominus w_x}$$

Exists, when x<0

$$\frac{F(w_x) \ominus F(w_y)}{w_x \ominus w_y}$$

Exists.
And suppose for |y| < |x|, xy ≥ 0, F (wx) ⊖ F (wy) exists, and when x > 0,

$$\frac{F(w_x) \ominus F(w_y)}{w_x \ominus w_y}$$

Exists, when x<0

$$\frac{F(w_y) \ominus F(w_z)}{w_y \ominus w_x}$$

Exists.
If there is a F(wx) ∈ E such that when x > 0

$$\lim_{y\to x^+} D\left(\frac{F(w_y)\ominus F(w_z)}{w_y\ominus w_x}, \acute{F}(w_x)\right)=0$$

and

$$\lim_{y\to x^-} D\left(\frac{F(w_x)\ominus F(w_y)}{w_x\ominus w_y}, \acute{F}(w_x)\right)=0$$

Hold When x<0

$$\lim_{y\to x^+} D\left(\frac{F(w_x)\ominus F(w_y)}{w_x\ominus w_y}, \acute{F}(w_x)\right)=0$$

And

$$\lim_{y\to x^-} D\left(\frac{F(w_y)\ominus F(w_z)}{w_y\ominus w_x}, \acute{F}(w_x)\right)=0$$

Hold. Then we say F is fuzzy differentiable at w_x and its fuzzy derivative at w_x is $\acute{F}(w_x)$.

III. NUMERICAL METHOD OF FUZZY DIFFERENTIAL EQUATION WITH FUZZY INITIAL VALUES

In this section, we are going to study the differential equation with fully fuzzy initial value as
Y'= F (x, y)
y(a0) = A0, a0,b0 ∈ E
where F: [a0, b0] × E ⟶ E is such that a0, b0 ∈ E, a0 (0) < $\underline{b}_0$(0) and
D^2((x, y), (x, y)) < δ

(1) $\forall\varepsilon > 0, \exists\delta > 0$

$$\Rightarrow D\,(F\,(x, y), F\,(x, y)) < \varepsilon$$

$$(x, y), (x, y) \in [a_0, b_0] \times E$$

(2) ∃L > 0, D (F (x, y1), F (x, y2)) ≤ LD (y1, y2) x, y1, y2 ∈ E
For numerically solving equation (1.2), we approximate *y*(b0).
We define the metric distance in *E*2 as follows:
*D*2 ((a,*b*), (a′ , *b*′)) = *max{D*(a,*a*′), *D*(b,*b*′)*}* ; (a,*b*), (a′ , *b*′) ∈ *E*2 (1.3)

Definition **3.1.** *Let u*0, *v*0 ∈ *E*, $\overline{a_0}(0) < \underline{v_0}(0)$. The fuzzy number set
{wt ∈ *E*|*wt* = (1 −*t*) a0 + *t*b0, *t* ∈ [0, 1]*}*
Is called fuzzy internal $[a_0, b_0]$.

Definition 3.2.
Suppose $[a_0, b_0]$ is the fuzzy interval. If
P [0,1] = {x0 = 0, x1, ..., xn = 1|x0 < x1 < ... < xn}
denote the partition of [0, 1]. If
wi = (1 − xi) a0 + xi b0; i = 0, 1, . . . , n
then, by Theorem 2.2
w0 = a0 ≺ w1 ≺ · · · ≺ wn−1 ≺ wn = b0. p˜[a0 ,b0] = {w0 = a0 , w1, . . . ,wn−1 , wn = b0 }
is the partition of [a0, b0].

Theorem 3.1.
Let u ∈ E, m, n ∈ R > 0 then mu ⊖g nu = (m − n) u.
Proof:

$$\text{mu} \ominus_g \text{nu} = (m-n)\,u \leftrightarrow \begin{cases} (i)ma = na + (m-n)a \\ \text{or} \\ (ii)na = ma + (-1)(m-n)a \end{cases}$$

(1.4)
(i)($m\underline{a}$,$m\overline{a}$) = ($n\underline{a}$,$n\overline{a}$)+ (m-n) (($\underline{a}$,$\overline{a}$)) if m≥n this case is correct.
(ii) ($n\underline{a}$,$n\overline{a}$) = ($m\underline{a}$,$m\overline{a}$) + (n-m) ($\underline{a}$,$\overline{a}$) if m<n this case is correct.

Example 3.1. Consider the following differential equation with fuzzy initial value.

$$\begin{cases} \acute{y} = xy \\ \underline{y(a_0)} = U_0 = (e^{0.005r^2+1}, e^{0.005r^2-0.02r+1.02}) \end{cases} \quad (1.5)$$

Where a_0 = (0.1r,0.2- 0.1r) and F(x,y) = xy. We want to approximate $y(b_0)$ by b_0 = (0.3+0.1r, 0.5-0.1r).

We show that F: $[a_0, b_0]$ x E→E satisfies in conditions (1) and (2).

Therefore (1) is satisfied if for fix $\epsilon>0$, $0<\delta<\frac{\epsilon}{|v_0|+M}$ exists such that

$M\geq \max_{x\epsilon[\underline{a_0}(0),\overline{b_0(0)}]}\{|y(x)|\}$

Assume

$D2((x, y), (x', y')) = \max\{D(x, x'), D(y, y')\} < \delta$.

Then

$D(F(x,y),F(x',y')) = \sup_{r\epsilon[0,1]}\max\{|\underline{xy}(r)-\underline{x'y'}(r)|, |\overline{xy}(r)-\overline{x'y'}(r)|\} = \Lambda$

i. If

$\Lambda = |\underline{x}(r*)\underline{y}(r*) - \underline{x}(r*)\underline{y}(r*)|$

$\leq |\underline{x}(r*)|D(y, y) + |\underline{y}(r*)|D(x, x)$

$< |v0(0)|\delta + M\delta$

$= (|v0(0)| + M)\delta$

$< \varepsilon$

ii. If $\Lambda = |x(r*)y(r*) - x'(r*)y'(r*)|$

$\leq |(r*)|D(y, y') + |y(r*)|D(x, x')$

$< |v0(0)|\delta + M\delta$

$= (|v0(0)| + M)\delta$

$< \varepsilon$

(2) is satisfied, since

$D(F(x, y), F(x, y')) = D(xy, xy') = \sup_{r\epsilon[0,1]}\max\{|\underline{xy}(r) - \underline{xy}(r)|, |xy(r) - xy(r)|\} = \Psi$

I. If,

$\Psi = |\overline{x}(r^*)\overline{y}(r^*) - \overline{x}(r^*)\overline{y}(r^*)| = |\overline{x}(r^*)||\overline{y}(r^*) - \overline{y}(r^*)| \leq LD(y,y')$

II. If,

$\Psi = |\underline{x}(r*)\underline{y}(r*) - \underline{x}(r*)\underline{y}(r*)| = |\underline{x}(r*)||\underline{y}(r*) - \underline{y}(r*)| \leq LD(y, y')$

We used n=4 with P [0,1] = $\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4}\}$ hence

$w0 = a0 = (0.1r, 0.2 - 0.1r)$

$w1 = \frac{3}{4}a0 + \frac{1}{4}b0 = (0.1r + 0.075, 0.275 - 0.1r)$

$w2 = \frac{1}{2}a0 + \frac{1}{2}b0 = (0.15 + 0.1r, 0.35 - 0.1r)$

$w3 = \frac{1}{4}a0 + \frac{3}{4}b0 = (0.225 + 0.1r, 0.425 - 0.1r)$

$w4 = b0 = (0.3 + 0.1r, 0.5 - 0.1r)$

$y(w_i+1) \simeq y(w_i) + (\frac{1}{4}b0 + (-\frac{1}{4})a0)\, w_i\, y(w_i)$, i = 0, 1, 2, 3

By $\frac{1}{4}b0 + (-\frac{1}{4})a0 = (0.025 + 0.05r, 0.125 - 0.05r)$

$Y(w_i+1) \simeq y(w_i) + (0.025 + 0.05r, 0.125 - 0.05r)\, w_i\, y(w_i)$, $i = 0, 1, 2, 3$

$y(w_4) = y(b_0) \simeq y(w_3) + (0.025 + 0.05r, 0.125 - 0.05r)\, w_3 y(w_3)$

If we used of n=8 with p [0,1] = $\{0,\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{4}{8},\frac{5}{8},\frac{6}{8},\frac{7}{8},\frac{8}{8}\}$ then

$w0 = u0 = (0.1r, 0.2 - 0.1r)$, $w1 = (0.0375 + 0.1r, 0.2375 - 0.1r)$

$w2 = (0.075 + 0.1r, 0.275 - 0.1r)$, $w3 = (0.1125 + 0.1r, 0.3125 - 0.1r)$

$w4 = (0.15 + 0.1r, 0.35 - 0.1r)$, $w5 = (0.1875 + 0.1r, 0.3875 - 0.1r)$

$w6 = (0.225 + 0.1r, 0.425 - 0.1r)$, $w7 = (0.2625 + 0.1r, 0.4625 - 0.1r)$

$w8 = v0 = (0.3 + 0.1r, 0.5 - 0.1r)$

$y(w_i+1) \simeq y(w_i) + (\frac{1}{8}v_0 + (-\frac{1}{8})a_0)w_i\, y(w_i)$, $i = 0, 1, .., 7$

By $\frac{1}{8}b_0 + (-\frac{1}{8})b_0$ = (0.0125 + 0.025r, 0.0625 − 0.025r)

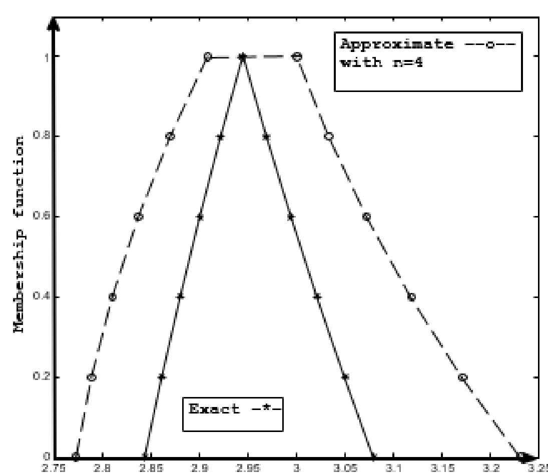
$y(w_i+1) \simeq y(w_i)$ + (0.0125 + 0.025r, 0.0625 − 0.025r)$w_i\, y(w_i)$, i = 0, 1, . . . , 7

$y(w8) = y(b_0) \simeq y(w7)$ + (0.0125 + 0.025r, 0.0625 − 0.025r) w7y(w7)

The exact solution of (3.6) is $y = e^{\frac{x^2}{2}+1}$ therefore.

$$y(b_0) = e^{\frac{x^2}{2}+1}$$
$$= (e^{0.005r^2+0.03r+1.045}, e^{0.005r^2-0.05r+1.125})$$

Figure 1 shows the comparison of the exact and approximated solutions for n = 4, 8.



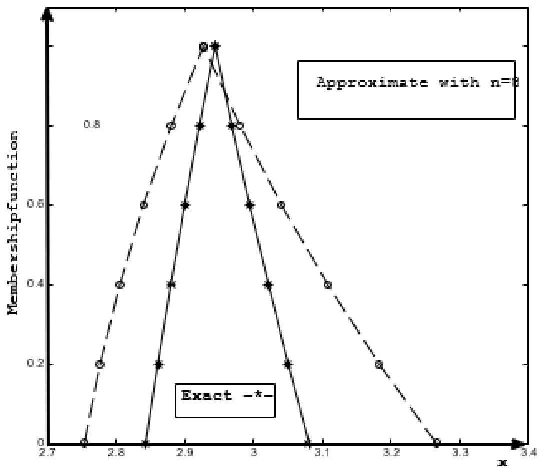


Figure 1: Comparison of the exact and approximated solutions for n = 4, 8.

IV CONCLUSION

In this paper, we studied differential equation with fully fuzzy values. We reviewed the partition of fuzzy interval and generalization of Hukuhara difference and division. Then, we solved the mentioned equation using our proposed numerical method and provided an example to illustrate this method.

REFERENCE

[1] S. Abbasbandy, T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computational Methods in Applied Mathematics 2 (2002) 113-124.

[2] S. Abbasbandy, T. Allahviranloo, Oscar Lopez-Pouso, Juan J. Nieto, Numerical meth- ods for fuzzy differential inclusions, Journal of Computer and Mathematics with Applications 48 (2004) 1633-1641.

[3] S. Abbasbandy, T. Allah Viranloo, Numerical solution of fuzzy differential equation by RungeKutta method, Nonlinear Studies 11 (2004) 117-129.

[4] S. Abbasbandy, M. Otadi, M. Mosleh, Minimal solution of general dual fuzzy linear systems, Chaos, Solitons and Fractals 37 (2008) 1113 - 1124.

[5] T. Allahviranloo, Successive over relaxation iterative method for fuzzy system of linear equations, Applied Mathematics and Computation 162 (2005) 189-196

[6] T. Allahviranloo, N.A. Kiani, M. Barkhordari, Toward the existence and uniqueness of solutions of second-order fuzzy differential equations, Information Sciences, Volume 179, Issue 8, 29 (2009) 1207-1215.

[7] T. Allahviranloo, N.A. Kiani, N. Motamedi, Solving fuzzy differential equations by differential transformation method, Information Sciences, Volume 179, Issue 7, 15 (2009) 956-96.

[8] T. Allahviranloo, N. Ahmady, E. Ahmady, Numerical solution of fuzzy differential equations by predictor corrector method, Information Sciences 177 (2007) 1633-1647.

[9] S.L. Chang, L.A. Zadeh, On fuzzy mapping and control, IEEE Transaction on Systems Man Cybernetics 2 (1972) 30-34.

[10] D. Dubois, H. Prade, Towards fuzzy differential calculus: Part 3, Differentiation, Fuzzy Sets and Systems 8 (1982) 225-233.

[11] W. Fei, Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients, Information Sciences 177 (2007) 4329-4337.

[12] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.

[13] H. Li, C. Wu, The integral of fuzzy mapping over a directed line, Fuzzy Sets and Systems 158 (2007) 2317-2338.

[14] G. Papaschinopoulos, G. Stefanidou, P. Efraimidi, Existence, uniqueness and asymp- totic behavior of the solutions of a fuzzy differential equation with piecewise constant argument, Information Sciences 177 (2007) 3855-3870. http://dx.doi.org/10.1016/j.ins.2007.03.006.

[15] J. Park, Intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals 22 (2004)1039-1046. http://dx.doi.org/10.1016/j.chaos.2004.02.051.

[16] S. Pederson, M. Sambandham, Numerical solution of hybrid fuzzy differential equation IVPs by a characterization theorem, Information Sciences 179 (2009) 319-328.

[17] M.L. Puri, D.A. Ralescu, Differentials of fuzzy function, Jornal of Mathematical Anal- ysis And Applications 91 (1983) 552-558.

[18] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330.

[19] P. Sevastjanov, L. Dymova, A new method for solving interval and fuzzy equations: Linear case,

Information Sciences 179 (2009) 925-937. http://dx.doi.org/10.1016/j.ins.2008.11.031.

[20] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems 161 (2010) 1564-1584. http://dx.doi.org/10.1016/j.fss.2009.06.009.

[21] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets and Systems 120 (2001) 523-532. http://dx.doi.org/10.1016/S0165-0114(99)00057-3.