Integral Solution of the Biquadratic Equation with Five Unknowns

$$(x+y)^2+xy+(z+w)^2-zw=(5a^2+3b^2)P^4$$

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Abstract **- We obtain infinitely many non-zero integer quintuples (x, y, z, w, P) satisfying the Biquadratic equation with five unknowns**

$$(x+y)^2+xy+(z+w)^2-zw=(5a^2+3b^2)P^4$$

Different approaches for finding the solution to the given equation are obtained.

Index Terms **- Biquadratic equation with five unknowns, Integral solutions.**

MSC 2000 Mathematics subject classification: 11D25

Notations:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

OH_n - Octahedral number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,6}$ - Four dimensional hexagonal number of rank n

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. In this context one may refer [4 -11] for various problems on the biquadratic diophantine equations with four and five variables. This paper concerns with yet another problem of determining non-trivial integral solutions of the non-homogeneous biquadratic equation with five unknowns given by

$$(x+y)^2+xy+(z+w)^2-zw=(5a^2+3b^2)P^4$$

A few relations among the solutions are presented.

II. METHOD OF ANALYSIS

The Diophantine equation representing the biquadratic equation under consideration with five unknowns is given by

$$(x+y)^2+xy+(z+w)^2-zw=(5a^2+3b^2)P^4 \quad (1)$$

Introducing the linear transformations

$$x=au+\sigma,\ y=au-\sigma,\ \ z=bv+\sigma,\ w=bv-\sigma,\ \sigma\neq 0 \quad (2)$$

in (1) it simplifies to

$$5a^2u^2+3b^2v^2=(5a^2+3b^2)P^4 \quad (3)$$

Again using the transformation

$$u=X+3b^2T, v=X-5a^2T \quad (4)$$

in (3),it simplifies to

$$X^2+15(abT)^2=P^4 \quad (5)$$

The above equation (5) is solved through different approaches and thus, one obtains distinct sets of solutions to (1)

A. *Approach1:*

Let $P=\alpha^2+15(ab\beta)^2$ (6)

Substituting (6) in (5) and using the method of factorisation, define

$$(X+i\sqrt{15}abT)=(\alpha+i\sqrt{15}ab\beta)^4 \quad (7)$$

Equating real and imaginary parts in (7) we get,

$$X=\alpha^4-90\alpha^2a^2b^2\beta^2+15^2(ab)^4\beta^4 \quad (8)$$

$$T=4\alpha^3\beta-60\alpha a^2b^2\beta^3 \quad (9)$$

In view of (8), (9), (4) and (2), the integral solution to (1) is obtained as follows:

$$\left.\begin{aligned} x&=a(f+3b^2g)+\sigma\\ y&=a(f+3b^2g)-\sigma\\ z&=b(f-5a^2g)+\sigma\\ w&=b(f-5a^2g)-\sigma \end{aligned}\right\} \quad (10)$$

where

$$\left.\begin{aligned} f(\alpha,\beta)&=\alpha^4-90\alpha^2a^2b^2\beta^2+15^2(ab)^4\beta^4\\ g(\alpha,\beta)&=4\alpha^3\beta-60\alpha a^2b^2\beta^3 \end{aligned}\right\} \quad (11)$$

Properties:

1. $3\sigma[x(\alpha,\beta)-y(\alpha,\beta)]$ is a nasty number.
2. $2\sigma^2[x(\alpha,\beta)-y(\alpha,\beta)+z(\alpha,\beta)-w(\alpha,\beta)]$ is a cubic integer.
3. $3P(\alpha,\alpha)=(1+15ab)(6P_\alpha^3+CP_{\alpha,6}-4P_\alpha^5]$
4. $x(\alpha,1)-y(\alpha,1)=2a[T_{4,\alpha}^2+15^2(ab)^4+12b^2(CP_{\alpha,6})-90(ab)^2(PR_\alpha-2T_{3,\alpha}+T_{4,\alpha})+180a^2b^4(SO_\alpha-2CP_{\alpha,6})]$
5. $z(\alpha,1)+w(\alpha,1)=2b[T_{4,\alpha}^2+15^2(ab)^4-20a^2(CP_{\alpha,6})-90(ab)^2(2P_\alpha^5-CP_{\alpha,6})+300a^4b^2(3(OH_\alpha)-2CP_{\alpha,6})]$
6. $x(p,p)+y(p,p)+z(p,p)+w(p,p)=2[a-90a^3b^2+15a^5b^4+ab^2-60a^3b^4][6F_{4,p,6}-6P_p^5+T_{4,p}]$
7. $5ax(\alpha,1)+3by(\alpha,1)-(10a^2+5b^2)[T_{4,\alpha}^2-45a^2b^2(T_{6,\alpha}-T_{4,\alpha}+2T_{3,\alpha})]\equiv 0(\text{mod }225)$

B. Approach2:

Write (5) as,

$$X^2+15(abT)^2=P^4*1 \quad (12)$$

(i) Take 1 as,

$$1=\frac{(5a^2-3b^2+i2ab\sqrt{15})(5a^2-3b^2-i2ab\sqrt{15})}{(5a^2+3b^2)^2} \quad (13)$$

Substituting (13) and (6) in (12) and using the method of factorisation, define,

$$(X + i\sqrt{15}abT) = \frac{(5a^2 - 3b^2 + i2ab\sqrt{15})}{(5a^2 + 3b^2)}(\alpha + i\sqrt{15}ab\beta)^4 \tag{14}$$

Equating real and imaginary parts in (14) we obtain,

$$\left.\begin{aligned} X &= \frac{1}{(5a^2 + 3b^2)}\left[(5a^2 - 3b^2)f(\alpha,\beta) - 30(ab)^2.g(\alpha,\beta)\right] \\ T &= \frac{1}{(5a^2 + 3b^2)}\left[(5a^2 - 3b^2)g(\alpha,\beta) + 2f(\alpha,\beta)\right] \end{aligned}\right\} \tag{15}$$

In view of (2), (4) & (15) and making some algebra, the corresponding values of x, y, z, w and P are obtained as,

$$x = a(5a^2 + 3b^2)\left[\begin{aligned} &\{(5a^2 - 3b^2)f(\alpha,\beta) - 30(ab)^2.g(\alpha,\beta)\} + \\ &\qquad 3b^2\{(5a^2 - 3b^2)g(\alpha,\beta) + 2f(\alpha,\beta)\} \end{aligned}\right] + \sigma$$

$$y = a(5a^2 + 3b^2)\left[\begin{aligned} &\{(5a^2 - 3b^2)f(\alpha,\beta) - 30(ab)^2.g(\alpha,\beta)\} + \\ &\qquad 3b^2\{(5a^2 - 3b^2)g(\alpha,\beta) + 2f(\alpha,\beta)\} \end{aligned}\right] - \sigma$$

$$z = b(5a^2 + 3b^2)\left[\begin{aligned} &\{(5a^2 - 3b^2)f(\alpha,\beta) - 30(ab)^2.g(\alpha,\beta)\} - \\ &\qquad 5a^2\{(5a^2 - 3b^2)g(\alpha,\beta) + 2f(\alpha,\beta)\} \end{aligned}\right] + \sigma$$

$$w = b(5a^2 + 3b^2)\left[\begin{aligned} &\{(5a^2 - 3b^2)f(\alpha,\beta) - 30(ab)^2.g(\alpha,\beta)\} - \\ &\qquad 5a^2\{(5a^2 - 3b^2)g(\alpha,\beta) + 2f(\alpha,\beta)\} \end{aligned}\right] - \sigma$$

$$P = (5a^2 + 3b^2)^2(\alpha^2 + 15(ab\beta)^2)$$

(ii) 1 can also be taken as

$$1 = \frac{(\delta^2 - 15a^2b^2 + i2ab\delta\sqrt{15})(\delta^2 - 15a^2b^2 - i2ab\delta\sqrt{15})}{(\delta^2 + 15a^2b^2)^2} \tag{16}$$

Substituting (16) and (6) in (12) and using the same procedure as approch2, we can get another non-trivial integral solutions to (1)

(iii) Also 1 can be written as

$$1 = \frac{(1 + i\sqrt{15})(1 - i\sqrt{15})}{4^2} \tag{17}$$

Substituting (17) and (6) in (12) and using the method of factorisation, define,

$$(X + i\sqrt{15}abT) = 1 = \frac{(1 + i\sqrt{15})}{4}(\alpha + i\sqrt{15}ab\beta)^4 \tag{18}$$

Equating real and imaginary parts in (18) we get

$$\left.\begin{aligned} X &= \frac{1}{4}[f(\alpha,\beta) - 15g(\alpha,\beta)] \\ T &= \frac{1}{4}[g(\alpha,\beta) + f(\alpha,\beta)] \end{aligned}\right\} \quad (19)$$

In view of (2), (4), (11) and (19), the corresponding values of x, y, z, w, P can be obtained.

(iv) 1 can also be written as

$$1 = \frac{(\alpha - 15ab + i2\alpha ab\sqrt{15})(\alpha - 15ab - i2\alpha ab\sqrt{15})}{(\alpha + 15ab)^2}$$

Using the same procedure as above the solutions of (1) can be obtained.

C. Approach3:

The assumption, $X = X'P, T = T'P$ (20)

in (5) leads to the equation,

$$X'^2 + 15(abT')^2 = P^2 \quad (21)$$

Then the solution to (21) is

$$X' = r^2 - 15s^2, abT' = 2rs, P = r^2 + 15s^2 \quad (22)$$

In view of (22), (20), (4) and (2) the integral solution of (1) can be obtained as

$$x = a[(ab)^4(R^4 - 15^2S^4) + 6b^2(ab)^3(R^2 + 15S^2)RS] + \sigma$$
$$y = a[(ab)^4(R^4 - 15^2S^4) + 6b^2(ab)^3(R^2 + 15S^2)RS] - \sigma$$
$$z = b[(ab)^4(R^4 - 15^2S^4) - 10a^2(ab)^3(R^2 + 15S^2)RS] + \sigma$$
$$w = b[(ab)^4(R^4 - 15^2S^4) - 10a^2(ab)^3(R^2 + 15S^2)RS] - \sigma$$
$$P = P = (ab)^2(R^2 + 15S^2)$$

D. Approach4:

Arranging (21) as

$$P^2 - X'^2 = (5a^2T')(3b^2T') \quad (23)$$

and using the method of factorisation, writing (24) as a system of double equations,

we get the solution of (23) as

$$\left.\begin{aligned} X' &= 2(5A^2 - 3B^2)T' \\ P &= 2(5A^2 + 3B^2)T' \end{aligned}\right\} \quad (24)$$

By taking $T' = 2k$ and using (24), (20), (4) and (2), the integral solution can be obtained as follows:

$$x = a[k^2(25A^4 - 9B^4) + 6b^2k(5A^2 + 3B^2)] + \sigma$$
$$y = a[k^2(25A^4 - 9B^4) + 6b^2k(5A^2 + 3B^2)] - \sigma$$
$$z = b[k^2(25A^4 - 9B^4) - 10a^2k(5A^2 + 3B^2)] + \sigma$$
$$w = a[k^2(25A^4 - 9B^4) - 10a^2k(5A^2 + 3B^2)] - \sigma$$
$$P = (5A^2 + 3B^2)k$$

E. Approach5:

Writing (23) as a system of double equations in a different manner as

$$(P + X')(P + X') = (5abT')(3abT'),$$

and solving we get the solution of (21) as

$$\left.\begin{aligned} X' &= abT' \\ P &= 4abT' \end{aligned}\right\} \quad (25)$$

Taking $T' = k$ and using (25), (20), (4) & (2), we get the corresponding integral solution of (1) as follows:

$$x = a[4a^2b^2k^2 + 12ab^3k^2] + \sigma$$
$$y = a[4a^2b^2k^2 + 12ab^3k^2] - \sigma$$
$$z = b[4a^2b^2k^2 - 20a^3bk] + \sigma$$
$$w = b[4a^2b^2k^2 - 20a^3bk] - \sigma$$
$$P = 4abk$$

III. CONCLUSION

Instead of (4), the introduction of the transformations

$$u = X - 3b^2T, v = X + 5a^2T$$

in (3) leads to (5).By a similar procedure we can obtain different pattern of integral solutions to (1) and their corresponding properties.

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