An Heptagonal Numbers

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***Abstract* - Two Results of interest (i)There exists an infinite pairs of heptagonal numbers (H_m , H_k) such that their ratio is equal to a non –zero square-free integer and (ii)the general form of the rank of square heptagonal number (H_m) is given by m=$\frac{3}{20}$ [$\left(19+3\sqrt{40}\right)^{2r+1}+\left(19-3\sqrt{40}\right)^{2r+1}$+2], where r = 0,1,2…….relating to heptagonal number are presented. A Few Relations among heptagonal and triangular number are given.**

***Index Terms* - Infinite pairs of heptagonal number, the rank of square heptagonal numbers, square-free integer.**

I. PRELIMINARIES

Definition(1.1):
A number is a count or measurement heptagon:
A heptagon is a seven –sided polygon. It also has seven vertices, or corners where sides meet and seven angles.

Definition(1.2):
Heptagonal numbers:
A Heptagonal number is a figurate number that represents a heptagon number is given by the formula$\frac{5n^2-3n}{2}$
The first few heptagonal numbers are 1,18,34,55,81,112,148,189,235,286,342,403,469,540, 616,697,783,874,970,1071,1177,1288,1404,1525,165 1,1782…………

Definition(1.3):
Square numbers:
A perfect square is number that can be expressed as the product of two equal integer for example 16,25,36….are perfect squares.

Definition (1.4):
Square –free numbers:
A number is said to be square free if no prime factor divides it more than once, that is largest power of a prime factor that divides n is one. First few square free number are 1,2,3,5,6,7,10,11,13,14,15,17……

Definition(1.5):
Square free integer:
A Square – free Integer is an integer which is divisible by no perfect Square other than 1. That is, its prime factorization has exactly one factors for each prime that appears in it. For example
10 =2.5 is square free,
But 18=2.3.3 is not because 18 is divisible by 9=32
The smallest positive square free numbers are 1,2,3,5,6,7,10,11,13,14,15,17,19,21,22,23…….

Definition(1.6):
Sequences of Heptagonal numbers:
The sequences of heptagonal numbers are
$n \in z \geq 0$
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0,1,7,18,34,55,81,112,148,189,235,286,342,403,469, 540. Heptagonal number us heptagon number.

II-INTRODUCTION

Heptagonal number have fascinated mathematicians as they provide non-routine and challenging problems. For a review on heptagonal numbers and their properties one may refer [1],[2]. In this communication we present two results of interest relating to heptagonal numbers. Also a few relations among heptagonal and triangular numbers are presented.
Result I: There exists an infinite pairs of heptagonal numbers (H_m , H_k) such that their ratio is equal to a non –zero square free integer.
Proof:
let(H_m , H_k) be a pair of heptagonal numbers such that

$$\frac{H_m}{H_k} = \propto \cdots \dots \dots \dots \dots \dots . . (1)$$

Where $\propto$ is a non –zero square-free integer. using the standard definition on heptagonal numbers, equation (1)is written as

Y^2- $\propto X^2 = -N$ … … … … … … … (2)

Where N=9($\propto -1$),

X=10K-3,

Y=10m-3……………………………..(3)

The above equation (2) is well known, and it has an infinite number of integral solutions. The general form of an integral solution of the equation (1) is represented by

$Y_n = \frac{1}{2}[(Y_0 + \sqrt{\propto} X_0)^{n+1}(Y_0 + \sqrt{\propto} X_0) + (Y_0 - \sqrt{\propto} X_0)^{n+1}(Y_0 - \sqrt{\propto} X_0)]$,

$X_n = \frac{1}{2\sqrt{\propto}}[(Y_0 + \sqrt{\propto} X_0)^{n+1}(Y_0 + \sqrt{\propto} X_0) + (Y_0 - \sqrt{\propto} X_0)^{n+1}(X_0\sqrt{\propto} - Y_0)]$,

Where $(Y_0 - \sqrt{\propto} X_0)$ *is the fundamental solution of equation* (1) and $(Y_0 - \sqrt{\propto} X_0)$ is the fundamental solution of Y^2- $\propto X^2 = 1$, In view of equation (3), the ranks of the respective heptagonal numbers are given by

m=$\frac{1}{20}[(Y_0 - \sqrt{\propto} X_0)^{n+1}(Y_0 + \sqrt{\propto} X_0) + (Y_0 - \sqrt{\propto} X_0)^{n+1}(Y_0 - \sqrt{\propto} X_0) + 6]$,

$X_n = \frac{1}{20\sqrt{\propto}}[(Y_0 + \sqrt{\propto} X_0)^{n+1}(Y_0 + \sqrt{\propto} X_0) + (Y_0 - \sqrt{\propto} X_0)^{n+1}(X_0\sqrt{\propto} - Y_0) + 6\sqrt{\propto}]$,

Where n=0,1,2,3,…..for the sake of simplicity a few heptagonal numbers with their corresponding ranks when $\propto$ 2,6,7. *are presented in the table I below*

Values of $\propto$	Rank m	Rank k	Heptagonal number H_m	Heptagonal number H_k
2	72	51	12852	6426
2	2810796	1987533	19751431167846	9875715583923
2	1101188812960	77915256855	3035393625366937668456 0	1517696812686468834228 0
6	51	21	6426	1071
6	486843	198753	592539536358	98756589393
7	2214	837	122251169	1750167
7	36274357737	13710418506	328957257302521786631 7	469938939003602552331

RESULT 2:

The general form of the rank of square heptagonal numbers (H_m) is given

m=$\frac{3}{20}[(19 + 3\sqrt{40})^{2r+1}$+$(19 - 3\sqrt{40})^{2r+1} + 2]$,

where r=0,1,2,…..

Proof:

Let H_m be a square heptagonal number, We write

$$H_m = t^2 \text{ … … … … … … … … (2.1)}$$

Where 't' is a non –zero integer. using the definition of heptagonal number the above equation(2.1)is written as

$y^2 = 40t^2 = 9$ … … … … … … … .. (2.2) where

y=10m-3…………………….(2.3)

The values of 't' and 'y' satisfying the equation (2.2) are given by

Y_n=$\frac{3}{2}[(19 + 3\sqrt{40})^{n+1}$+$(19 - 3\sqrt{40})^{n+1}]$,

t_n=$\frac{3}{2}[\sqrt{40}(19 + 3\sqrt{40})^{n+1}$+$(19 - 3\sqrt{40})^{n+1}]$,

In view of the equation (2.3), the rank m is given by

m =$\frac{3}{20}[(19 + 3\sqrt{40})^{n+1}$+ $(19 - 3\sqrt{40})^{n+1} + 2]$,

It is noted that the values of m will be an integer when n is taking even values. Thus the rank m of the square heptagonal number H_m is given by

m=$\frac{3}{20}[(19 + 3\sqrt{40})^{2r+1}$+$(19 - 3\sqrt{40})^{2r+1} + 2]$,

where r=0,1,2,…..

some example ate presented in the Table -2

Values of r	Rank m	Heptagonal number H_m
0	6	81
1	821	168662169
2	11844150	350709705290025
3	17079255654	729252434311108535809
4	24628274808486	1516379800105728351531817761

The rank $\boldsymbol{m}_r$satisfies the recurrence relation $\boldsymbol{m}_{r+2} - 1442m_{r+1} + m_r + 432 = 0$.

A Few Interesting Relation Among Heptagonal Numbers:

Let H_m *and* T_S denote the heptagonal and triangular numbers respectively for the sake of simplicity and brevity we present below a few interesting relations

(i)$H_{m+1} + H_{m-1} - 2H_m = 5$ (Recurrence relation)

(ii) $\sum_{m=1}^{s} H_m = \frac{T_S}{3}(5S + 2)$

(iii)$\sum_{m=1}^{5} H_m^2 = \frac{T_s}{6}[15s^3 - 11s + 2]$

(iv)$\sum_{m=1}^{5} H_m^3 = \frac{T_S}{336}[1550S^5 - 2775S^4 + 63240S^3 + 101568S^2 + 8948S - 11340]$

III.CONCLUSION

One may search for the other infinite pairs of heptagonal numbers. whose ratios are non –square free integers and the other ranks of square heptagonal numbers.

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