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Evaluation All Possible Subgroups of a Group of Order 30 By Using Sylow's Theorem

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Abstract: In this work, we will discuss the concept of group, order of a group, along with the associated notions of automorphisms group of the dihedral groups and split extensions of groups. This workis a generalization of the Sylow's Theorems. Then we find all the groups of order 30 which will give us a practical knowledge to see the applications of the Sylow's Theorems. For this, we also have used some known results of Semi-direct Product of groups, some important definitions as like as the exact sequences and split extensions of groups and P - Sylow's Theorem to obtain our result. Finally, we have found all subgroups of group order 30 forAbelian and Non-abelian cases.

Index Terms: Dihedral group, exact sequences, split extensions of groups, Lagrange's Theorems and P-Sylow's Theorems.

INTRODUCTION

It's not true for any number dividing the order of a group, there exists a subgroup of that order. For example, the group S 4 of even permutations on the set {1, 2, 3, 4} has order 12, yet there does not exist a subgroup of order 6. As usual we can use Lagrange's Theorem to evaluate subgroups of group of different orders such as order 2, 4, 6,8, 9,10,12, etc., i.e. whose order not so high (Not higher order groups). But it is not possible to evaluate subgroups of higher order group as like as 30, 35, 40, 45, 50, etc. by using Lagrange's Theorems. For this case, applying P-Sylow's Theorems we can easily evaluate all possible subgroups of any higher order groups. The Sylow's Theorems is very important part of finite group theory and the classification of finite simple groups [1, 3]. The order of sylow's P -subgroup of a finite group G is Pn, where n is the multiplicity of P in the order of G and any subgroup of order P n subgroup of G.

Preliminaries:
Dihedral group: A dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. Dihedral groups are among the simplest examples of finite groups, and they play an important role in group theory, geometry, and chemistry.

Notation of dihedral group:

$$D = \langle a, b : a^n = b^2 = (ab)^2 = I \rangle$$

Definition of P-group:
When p is a prime number, then a p-group is a group, all of whose elements have order some
power of p. For a finite group, the equivalent definition is that the number of elements in Gis a power of p. In fact, every finite group has subgroups which are p-groups by the Sylow's theorems, in which case they are called Sylow p-subgroups.

Definition of Sylow P-subgroup:
If pk is the highest power of a prime p dividing the order of a finite group G, then a subgroup of G of order pk is called a Sylow p-subgroup of G.
Index of a group:
Let G be a group. Let H be a subgroup of G.

The index [G: H] of H in G is the number of left (or right) cosets of G modulo H, or the number of elements in the left (or right) coset space G / H.

Lagrange's Theorem:
The order of each subgroup of a finite group, is a divisor of the order of the group, Such that

$$\frac{\circ(G)}{\circ(H)} = k$$

i. e the order of H is a divisor of order of G.
Sylow's First Theorem:
Let G be a finite group and p be a prime number. If m is the largest non-negative integer such that p^m is a divisor of ∘ (G), then G has a subgroup of order p^m

Sylow's second theorem:
Let G be a finite group and let p be a prime number such that p is a divisor of ∘ (G). Then, all sylow p-subgroups of G are conjugates of one another.

Sylow's third theorem:
Let G be a finite group and p be a prime number such that p ∘ (G). Then the number of sylow p-subgroups is of the form 1+ mp, where m is some non-negative integer.

Sylow's Fourth Theorem:
The number of Sylow p-subgroups of a finite group is congruent to 1(mod p).

Sylow's Fifth Theorem:
The number of Sylow p-subgroups of a finite groups is a divisor of their common index.

Automorphisms group of the dihedral group D_4:

Let $D_4 = \{e, x, x^2, x^3, y, yx, yx^2, yx^3\}$ with the defining relation $x^4 = y^2 = e, \quad y^{-1}xy = x^{-1}$, be the dihedral group of order 8.

Now, the conjugate classes of D_4 are:

$\{e\}, \{x^2\}, \{x, x^3\}, \{y, yx, yx^2, yx^3\}$.

So, $D_4 / \{e, x^2\} \cong$ to a group of order 4. So, D_4 has 4 inner automorphisms one of which is the identity. Then, let the other 3 inner automorphisms be α, β, γ. Now, if x is fixed by α then $\alpha(e) = e$, $\alpha(x) = x$ and $\alpha(y) = y, yx, yx^2, or \quad yx^3$. But $\alpha(y) \neq y$, for if $\alpha(y) = y$ then $\alpha = Id$,

which is not possible. Then, let $\alpha(y) = yx^2$ and hence $\alpha(yx) = \alpha(y)\alpha(x) = yx^3$ and therefore, $\alpha^2 = Id$. *Next*, if y is fixed by β then $\beta(e) = e$, $\beta(y) = y$ and $\beta(x) = x^{-1}$

$\beta(yx) = \beta(y)\beta(x) = yx^{-1}$ and $\beta^2 = Id$. Then $\gamma(e) = e$ and $\gamma(yx) = yx$ and $\gamma(x) = x^{-1}$, $\gamma(y) = yx^2$ and $\gamma^2 = Id$. Hence, we have $\gamma^2 = \beta^2 = \alpha^2 = Id$ and also, we have $\alpha\beta = \beta\alpha = \gamma$ and $\alpha\gamma = \gamma\alpha$.

Therefore inner $Aut(D_4) = \{Id, \alpha, \beta, \beta\alpha\} \cong C_2 \times C_2$ with $\alpha^2 = \beta^2 = Id$ and $\alpha\beta = \beta\alpha$.

Now, we consider the mapping $f: D_4 \to D_4$ With $f(e)=e$ and $f(x)=x$ or x^3. So, let $f(x)=x$ and assume that $g(x)=\alpha\ f(x)$then $g(x)=\alpha(x)=x$ and $g(y)\neq x^2$ for x^2 is a central element and hence $g(y)=y,\ yx,\ yx^2, or\ \ yx^3$.

If $g(y) = y$, then $g = Id$, and hence $g(y)\neq y$

If $g(y)=yx^2$ then $g=\alpha$ and hence $g(y)\neq yx^2$.

If $g(y)=yx$ then $g(yx)= yx^2$ and $g^4 = Id$.

Then, we have, $\beta g \beta = g^{-1}$ with $g^4 = \beta^2 = Id$, *and* also $\gamma g \gamma = g^{-1}$, with $g^4 = \gamma^2 = Id$.

Therefore, $Aut(D_4) \cong \{\beta, g\}$ with $g^4 = \beta^2 = Id$ and $\beta^{-1} g \beta = g^{-1}$.

Automorphisms group of the dihedral group D_6:

Let $D_6 = \{e, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$

With defining relation $x^6 = y^2 = e$ and $y^{-1}xy = x^{-1}$, be a dihedral group of order 12.

Now, the conjugate classes are:
$\{e\}, \{x, x^5\}, \{x^2, x^4\}\ \ \{x^3\}, \{y, yx^2, yx^4\}, \{yx, yx^3, yx^5\}$.

So, $D_6 | \{e, x^3\} \cong$ to a group of order 6. Then D_6 has 6 inner automorphisms one of which is the identity. Let the other inner automorphisms be Y, Z, U, V, T. *Now*, if x is fixed by Y, then $Y(e)=e$, $Z(x)=x$ and $Y(y)=yx^2$ and hence $Y(yx)=yx^5$. Then $Y^3 = Id$. Next, if y is fixed by Z then $Z(e)=e$, $Z(y)=$ y, and $Z(x)=x^{-1}$ and $Z(yx)=yx^{-1}$ and then $Z^2 =$ Id. Next, if yx is fixed by U then $U(e)=e$, $U(yx)=yx$, and $U(x)=x^{-1}$ and $U(y)=yx^2$. Lastly, if yx^5 is fixed by T then $T(e)=e$, $T(yx^5)=yx^5$ and $T(x)=x^{-1}$ and $T(yx)=yx^3$ and then $T^2 = Id$.

and hence we have, $Y^3 = Z^2 = U^2 = V^3 = T^2 = Id$ and by calculation we have, $Y^2 = V,$

$TU = V = ZT,\ UT = Y = TZ$ and hence $Z^{-1}YZ = Y^{-1},\ U^{-1}VU = V^{-1}, T^{-1}YT = Y^{-1}.$

Therefore, inner $Aut(D_6) = \{Z, Y\} \cong D_3 \cong S_3$ with $Z^{-1}YZ = Y^{-1}$ and $Y^3 = Z^2 = Id.$

Now, consider the mapping $S: D_6 \to D_6$

Let $S(e) = e$ then $S(x) = x$ or x^5 and so let $S(x) = x^5$ and put $M = US$ then

$M(x) = US(x) = U(x^5) = x$

Now, $M(y) \neq x^3$ for x^3 is a central element.

ALL GROUP OF ORDER 30

Non – Abelian Case

We keep in mind that 30=2.3.5

2- Sylow Subgroups:

The number x of 2-Sylow subgroup of a group G Of order 30 is $x \equiv 1(\text{mod}\,ulo2)$, where x= 1, 3,5,15.

1, 2-Sylow subgroup:

It implies that there is a proper normal subgroup in G which may be called N of order 2. Therefore, $N \cong C_2$ If $N \cong C_2$, then the sequence of group extension $\{e\} \to N \to G \to C_{15} \to \{e\}$

but (2,15) =1, so the extension splits. Now, $Y: C_{15} \to Aut(C_2) \cong Id$ and (15,2) =1, Where Y is a

constant homomorphism and the relation are given $b^{-1}ab = a^{-1}$ which is a commutative case. bySo, we exclude this case.

3, 2-Sylow subgroup:

The group G as a permutation on the objects, namely is 3,2-Sylow subgroups. It is a transitive group then the mapping $Y: G \to S_3$ gives that $Y(G) = A_3$ or S_3

i) If $Y(G) = A_3$ then the order of N is 10 and hence $N = C_{10}$ or $N \cong C_5XC_2$ or $N \cong D_5$, now first

two case are exclude for 2-Sylow subgroups of C_{10} and C_5XC_2 are characteristics.

ii) If $Y(G) = S_3$ then the order of N is 5. If $N \cong C_5$ then there is a mapping Z such that

$Z : S_3, Aut(C_5) = C_4 = C_2 X C_2$. Hence $Z(S_3) = \{e\}$ or C_2 Now, the group extension is given by $\{e\} \to N = C_2 X C_2 \to G_1 \to S_3 \to \{e\}$ where the defining relations of S_3 are $a^2 = b^2 = e$ and $b^{-1}ab = a^{-1}$. If ($Z(S_3) = \{e\}$) then $\ker(Z) = S_3$ and so let $N = \{t, u\}$ and $G = \{c, d\}$, where c, d

are mapped respectively to a, b. Now $a^2 = b^2 = e^5$, and

$s^2 = b^j$ with j=0,1 If $s^2 = b$ the

$c^2 = e, a = c^{-1}ac, a = b^{-1}ab, b = c^{-1}bc, a = d^{-1}ad, d^{-1}bd = b$

$a^2 = s^3 = c^2, ac = ca, as = sa, s^{-1}cs = c^{-1}ab, s^2cs^2 = s^{-1}(s^{-1}cs) = ca^{2i}b^{2k} = c$

and $s^2 \in \{a, b\}$ comments with c This implies that $a^{2i}b^{2k} =$ with 2i=0(mod5)and

$d = a^i b, d^{-1}cd = c^{-1}a^i b^k, a^i, b^k \in C \times C \in Aut(C) = D$ Now there exists some S∈ G such that

2k=0(mod2) which implies that i=0, k=0,1. If k =o then $s^{-1}cs = c^{-1}$ and if k=1 then $s^{-1}cs = c^{-1}ba$.

Note that C_3 generated by a is central because it commutes with every element, put

$f = c^2d^2, f^7 = e, d^2 = f^3$ and $d^{-1}cd = c^{-1}$ and so {f, d} generates D_5 but C_3 and D_5 are normal subgroups and $C_3 \cap D_5 = \{e\}$ and hence $G \cong C_3 x D_5$ which is a non –abelian group of 30.

5-2 Sylow subgroup:

The normalized of 2-sylow subgroup $N(S_2)$ must have an invariant subgroup of order 2.Now, the order of $N(s_2) = 4$ and So, $N(s_2) \cong C_4$ or $C_2 \times C_2$ or D_2 but $N(s_2) \neq D_2$ because none of them can have an invariant subgroup of order 2. The possibilities are (1) $N(s_2) = C_5$ (2) $C_5 x \Delta_{15}$ and so by Burnside's Theorem normal 2-complement exists. This will be abelian,

15, 2-Sylow subgroups imply that the order of $N(S_2)$ is 2

Now, the normal 2- component N of order 15, given that $N \cong C_{15}$ then group extension is given by $\{e\} \to C_{15} \to D \to H \to \{e\}$, where $H \cong C_2$ But (2, 15) =1 and so extension splits.

b1. If $H \cong C_{15}$ then $Y : C_2 \to Aut(C_{15}) \cong C_2 \times C_3$ and $Y(C_2) = \{e\}$

b2. If $Y(C_2) = \{e\}$ then $D \cong C_{15} x C_2$, which is abelian case and hence we drop it

b3. If $Y(C_2) = C_2$ then Y(c)=Z

b4. If Y(c) =Z then $c^{-1}ac = a, c^{-1}bc = b, ab = ba^j$ and this gives that G≅ $G \cong D_5 \times C_3$, which has already been found.

3-Sylow Subgroup:

The number x of 3-Sylow subgroups of a group G of order 30 is x = 1(modulo 3); where $x=1$, 10.

1, 3-Sylow Subgroups:

Any C_3 is a normal subgroup of G.

The group extension is $\{e\} \to C_3 \to G \to H \to \{e\}$ and H is of order 10. But (3, 10) =1, so the extension splits. And so $Y : H \to Aut(C_3) \cong C_2$.

(a) Let $H = C_{10} \cong C_5 \times C_2$ then $Ker Y$ contains C_2 and it commutes with, C_3, C_5 ,...

This gives that $G \cong C_2 \times N$, where N is a non-abelian group of order 15, which has already been found.

(b) Let $H = D_5$ and D_5 has no quotient group of order 4 , so $Y(D_7)$ has order 1. If $Y(D_5)$ has order 1 then $G \cong D_5 \times C_3$, which has already been found.

5-Sylow Subgroups:

The number x of 7-sylow subgroups of a group G of order 42 is $x \equiv 1(modulo 5)$; where $x = 1$.

1, 5-Sylow Subgroups:

Any C_5 is a normal subgroup of G.

The group extension is $\{e\} \to C_5 \to G \to H \to \{e\}$ and H is of order 6. But (5,6)=1, so the extension splits. And so $Y : H \to Aut(C_5) \cong C_6$.

(a) Let $H = C_6 \cong C_3 \times C_2$ then

$Ker Y$ Contains C_2 and it commutes with, C_5, C_3 ,.... This gives that $G \cong C_2 \times N$, where N is a non-abelian group of Order 15, which has already been found.

(b) Let $H = D_3$ and D_3 has an element of order 5, so Y (D_3) has order 1. If $Y(D_3)$ has order 1

CONCLUSION

We have found all possible subgroups of group of order 42 by applying P -Sylow's Theorem, which will give us a practical knowledge to see the applications of the Sylow's Theorems.

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