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# Integral Representation of Simple Bessel Polynomial

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***Abstract* - In the present paper we have obtained simple generating relation, contour integral representation, single infinite integral representation, finite double integral representation, infinite single integral representation of polynomial. $f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$**

***Index Terms* - Bessel polynomial, Generating relation, Integral Representation.**

## I.INTRODUCTION

Krall and Frank[3, P.47] studied the simple Bessel polynomials defined as follows

$$f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right) \qquad (1)$$

## II. SIMPLE GENERATING RELATION

By (1), we have $f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k \left(-\frac{x}{2}\right)^k}{k!}$$

But $(-n)_k = \frac{(-1)^k n!}{(n-k)!} (0 \le k \le n)$

$= 0 k > n$

$$f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(n+1)_k \left(-\frac{x}{2}\right)^k}{k!} \; (2)$$

$$\sum_{n=0}^{\infty} \frac{{}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right) t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!} \frac{(n+1)_k \left(-\frac{x}{2}\right)^k}{k!} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{n!} \frac{(n+1)_k \left(-\frac{x}{2}\right)^k}{k!} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{(n+1)_k \left(-\frac{x}{2}\right)^k}{(n-k)!\;\; k!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k+1)_k \left(-\frac{x}{2}\right)^k}{k!} \frac{t^{n+k}}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k+1)_k \left(-\frac{xt}{2}\right)^k}{k!} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{xt}{2}\right)^k}{k!} \sum_{n=0}^{\infty} (n+k+1)_k \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^k}{k!} \frac{1}{(1-t)^{n+k+1}}, \quad \left[since \sum_{n=0}^{\infty} (n+k+1)_k \frac{t^n}{n!} = \frac{1}{(1-t)^{n+k+1}}\right]$$

$$= \frac{1}{(1-t)^{n+1}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{xt}{2(1-t)}\right)^k$$

$$= (1-t)^{-n-1} e^{\frac{xt}{2(1-t)}} \qquad (2)$$

By using Maclaurin's theorem,

$f(t) = \sum_0^{\infty} \frac{f^n(0)\;\; t^n}{n!}$, where, $f^n(0) = \left[\frac{d^n}{dt^n} f(t)\right]_{t=0}$

$$f^n(0) = \frac{n!}{2\pi i} \int \frac{f(t)}{t^{n+1}}\, dt, \;\forall\, n = 0,1,2,3, \ldots$$

If $\quad (1-t)^{-n-1} e^{\frac{xt}{2(1-t)}} = \sum_{k=0}^{\infty} \frac{{}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right) t^n}{n!}$

then, $f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$

$$= \frac{n!}{2\pi i} \int \frac{(1-t)^{-n-1} e^{\frac{xt}{2(1-t)}}}{t^{n+1}}\; dt$$

$$= \frac{n!}{2\pi i} \int \frac{e^{\frac{xt}{2(1-t)}}}{(1-t)^{n+1} t^{n+1}}\; dt \qquad (3)$$

## III. SINGLE INFINITE REPRESENTATION

From (1),we have

$f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k \left(-\frac{x}{2}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{\Gamma(n+k+1)}{\Gamma(n+1)} \left(-\frac{x}{2}\right)^k$$
$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{1}{\Gamma(n+1)} \left(-\frac{x}{2}\right)^k \Gamma(n+k+1)$$
$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{1}{\Gamma(n+1)} \left(-\frac{x}{2}\right)^k 2 \int_0^{\infty} e^{-t^2} t^{2(n+k+1)-1}\, dt$$
$$= 2 \sum_{k=0}^{\infty} \frac{(-n)_k}{k!\ \Gamma(n+1)} \left(-\frac{x}{2}\right)^k \int_0^{\infty} e^{-t^2} t^{(2n+2k+1)}\, dt$$
$$= \frac{2}{\Gamma(n+1)} \int_0^{\infty} e^{-t^2} t^{(2n+1)} \left( \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(-\frac{xt^2}{2}\right)^k \right) dt$$
$$= \frac{2}{\Gamma(n+1)} \int_0^{\infty} e^{-t^2} t^{(2n+1)}\, {}_1F_0[-n;-;-;-\frac{xt^2}{2}]\, dt (4)$$

## IV.FINITE DOUBLE INTEGRAL REPRESENTATION

From Srivastava and Karlsson[6], we have,

$$\iint_D u^{a-1} v^{b-1} (1-u-v)^{c-1} dudv = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

*where D is bounded by the lines* $u \geq 0, v \geq 0$ *and* $u + v \leq 1$.

From (1),

$f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k \left(-\frac{x}{2}\right)^k}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{\Gamma(n+k+1)}{\Gamma(n+1)} \left(-\frac{x}{2}\right)^k$$
$$= \frac{\Gamma(\alpha)}{\Gamma(n+1)\Gamma(b)\Gamma(\alpha-n-1-b)}$$
$$\sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{\Gamma(b)\Gamma(\alpha-n-1-b)\Gamma(n+k+1)(\alpha)_k}{\Gamma(\alpha+k)} \left(-\frac{x}{2}\right)^k$$
$$= \frac{\Gamma(\alpha)}{\Gamma(n+1)\Gamma(b)\Gamma(\alpha-n-1-b)} \times$$
$$\sum_{k=0}^{\infty} \frac{(-n)_k (\alpha)_k}{k!} \iint_D u^{n+k+1-1} v^{b-1} (1-u-v)^{\alpha-n-1-b-1} \left(\frac{-x}{2}\right)^k dudv$$
$$= \frac{\Gamma(\alpha)}{\Gamma(n+1)\Gamma(b)\Gamma(\alpha-n-1-b)} \times$$
$$\sum_{k=0}^{\infty} \frac{(-n)_k (\alpha)_k}{k!} \iint_D u^{n+k} v^{b-1} (1-u-v)^{\alpha-n-1-b-1} \left(\frac{-x}{2}\right)^k dudv$$
$$= \frac{\Gamma(\alpha)}{\Gamma(n+1)\Gamma(b)\Gamma(\alpha-n-1-b)} \times$$
$$\iint_D u^n v^{b-1} (1-u-v)^{\alpha-n-b-2} \left[ \sum_{k=0}^{\infty} \frac{(-n)_k (\alpha)_k}{k!} \left(\frac{-xu}{2}\right)^k \right] dudv$$
$$= \frac{\Gamma(\alpha)}{\Gamma(n+1)\Gamma(b)\Gamma(\alpha-n-1-b)} \times$$
$$\iint_D u^n v^{b-1} (1-u-v)^{\alpha-n-b-2}\, {}_2F_0\left(-n, \alpha; -; \frac{-xu}{2}\right) dudv$$

(5)

## V.INFINITE SINGLE REPRESENTATION

From (1), we have $f_n(x) = {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right)$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k \left(-\frac{x}{2}\right)^k}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \frac{\Gamma(n+k+1)}{\Gamma(n+1)} \left(-\frac{x}{2}\right)^k$$
$$= \frac{1}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(-\frac{x}{2}\right)^k \Gamma(n+k+1)$$
$$= \frac{1}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(-\frac{x}{2}\right)^k \left[ \int_0^{\infty} e^{-t} t^{n+k+1-1}\, dt \right]$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(-\frac{x}{2}\right)^k e^{-t} t^{n+k}\, dt$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} \left(-\frac{xt}{2}\right)^k e^{-t} t^n\, dt$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\ k!} \left(-\frac{xt}{2}\right)^k e^{-t} t^n\, dt$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!\ k!} \left(\frac{xt}{2}\right)^k e^{-t} t^n\, dt$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{xt}{2}\right)^k e^{-t} t^n\, dt$$
$$= \frac{1}{\Gamma(n+1)} \int_0^{\infty} \left(1 + \frac{xt}{2}\right)^n e^{-t} t^n\, dt$$
$$= \frac{1}{2^n\ \Gamma(n+1)} \int_0^{\infty} (2 + xt)^n e^{-t} t^n\, dt$$
$$= \frac{1}{2^n\ \Gamma(n+1)} \int_0^{\infty} (2t + xt^2)^n e^{-t}\ dt \qquad (6)$$

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