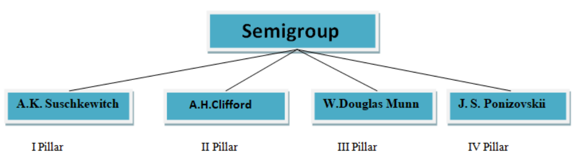
Background History of Semigroups in Algebra

Vasantha.G[1], Sri Lakshmi.T[2]
[1,2]*GITAM*

***Abstract*— This paper gathered information on the history of semigroup theory in algebra and added some basic definitions of Semigroup with possible examples. In our view, the father of the semigroup theory is Anton Kazimirovich Suschkewitch. Next, he developed many areas in the twentieth century. He proved every Semigroup might be contained in a complete transformation monoid after A.H Clifford worked on the same content as A.K. Suschkewitch. Later, he brought methods and results that are useful for coming researchers and studied the statement of Rees theorem found by David Ree. W.Douglas Munn has investigated semisimple semigroups and constructed an irreducible representation of 0-simple semigroups. J. S. Ponizovskii studied the Matrix representation of the Rees theorem. These four authors are the main four pillars of semigroup theory.**



Index Terms: **Semigroups, Simple semigroups, 0-simple.**

The subject of Classification: 20M10, 20M99

INTRODUCTION

This paper aims to give a historical survey of semigroup theory and basic definitions of what authors introduced terms. This paper studied the groundwork of characteristics and different kinds of semigroups. Early authors described the many types of semigroups and proved many methods. We investigated from roots of semigroups and worked on perplexing statements is completely regular semigroups. We found four main major authors' aspects of Semigroup theory. The first semigroup theorist in the world has Anton Kazimirovich Suschkewitch.In 1941 Clifford proved that" if S is a union of groups, then it is a semilattice of completely simple semigroups." Also, he proved that" A band is a semilattice of rectangular bands" [7]. But Inverse semigroup was first introduced by Vagner (1952-1953).

Ponizoivskii defined "Basic Semigroup" as follows. He implemented this definition in many proves.

Basic Semigroup:[6] Let" Ş<Ɱ(m,k) then Ş is basic if the following conditions hold.

1. W. L(S) =W. Where L(s) denotes a K-linear envelop of Ş.
2. If wϵW, then w.Ş=0, which implies w=0".

Note:" Poniziivskii proved every matrix of irreducible Semigroup is basic."[6]

Basic Definitions of semigroups:

Let Ş be a nonempty set with binary operations ʘ if the following postulates hold.

1) P,q ϵ Ş →pʘq ϵ Ş
2) P,q ,r ϵ Ş →(pʘq)ʘr=pʘ(qʘr)

Then (Ş,ʘ) is called Semigroup.

Example: Set of natural numbers is Semigroup under addition.

A semigroup Ş is regular. If an element p in Ş, then there exists aϵ Ş such that pap=p or apa=a.

Identity Element:

Let (Ş,*) be an algebraic structure and 'x' be an element of Ş. An element e_1 in Ş is said to be the left identity element concerning '*' if e_1 *x=x for all e_1 ϵ Ş

Similarly, an element e_2 in Ş is said to be the right identity element for '*' if x * e_2 =x for all e_2 ϵ Ş. The Left and right identities and unique, so we have e_1 =e_2 =e, which is called the identity element of Ş.

Monoid: An algebraic System (Ş,∗) is said to be a monoid if the following conditions are satisfied.

1. " ʘ" is a closed
2. " ʘ" is an associative
3. There is an identity in Ş.

A nonempty set T of a semigroup S is called a subsemigroup if it is closed and associative under the binary operation ′ʘ.'

that is 1.a ⊙ b∊T ∀ a,b∊T 2.a ⊙ (b ⊙ c)=(a ⊙ b)⊙ c ∀ a,b,c ∊T

Monogenic semigroups:
Let Ş be a semigroup, the order of the element p is defined as the order of the subsemigroup <p>. If S is a semigroup with an element an s.t =Ş<p>, then Ş is a monogenic semigroup.[1]

David Ree's (1940):
The first paper of David Ree on commutative algebra proves many theorems related to the matrix representation. His work on completely 0-simple semigroups along with Clifford's work in 1941. David gives a new proof of "Ore's theorem" [3], strikingly Semigroup theoretic.

Rees theorem: This Rees theorem was introduced by David Ree in 1940 as follows.
"Let G_0 be a 0-group, let K, H be nonempty sets, and let $P=(p_{\lambda i})$ be(KxH) matrix with entries in G_0. Suppose that P is regular in a sense and S=(KxGxH) U {0} and define a multiplication on S. Then S is a completely 0-Simple semigroup. Conversely, every completely 0-Simple Semigroup is isomorphic to one constructed in the way". [2]
David Ree developed the structure of semigroup theory in algebra and described classes of semigroups and 0-simple semigroups. He proved one famous theorem called Ree's theorem in 1940. He also worked on various types of classes of semigroups.

J.A. Green (1951):
From 1926-to 2014, James Alexander Green gave tremendous results and influential research work in his area. Worldwide everyone knows him as the name of Sandy Green. First, he described on Equivalence relation of semigroup theory. His famous described areas in semigroups as

- Free burnside semigroup.
- 0-bisimple semigroups.
- Minimal condition for left (right) ideals.
- Representation theory.

He described ideals on subsemigroups of semigroups from 1951.

A.K.SUSHKEVICH (1889-1961):

A.K.Sushkevich is the first pillar of Semigroup theory. He is one of the famous Russian mathematicians. He studied different classes of generalized semigroup theory and went to Berlin to study and also attended the lecturer of Frobenius. He submitted a Master's degree dissertation at the Kharkiv University in 1917. When he worked as Professor at the Voronezh University published many papers in the area of General theory of Groups in 1922.
He worked on

- Two-sided ideals of minimal semigroups.
- Simple Semigroups.
- Rees theorem.
- Special types of transformations and permutations.
- Modulo group theory.
- 0-Simple Semigroups.
- Embedded full transformation.

He showed that for any P in A, PA=A in general. Every finite semigroup Ş contains a minimal ideal I, Completely determined by

a) The structure of the abstract group C is isomorphic to the $C_{k\lambda}$,
b) The numbers p and q.
c) The (p-1)(q-1) products $E_{11}E_{k\lambda}$ (k=2,…….p) (λ=2,…….q) where $E_{k\lambda}$ denotes the identity of $C_{k\lambda}$.

One of his famous theorems is as follows

Theorem:
"All representation of an ordinary (finite) group using mxm matrices of rank n<m may be obtained from the representation of the same group by nxn matrices of rank n."[2]

A.H. Clifford (1908-1992):

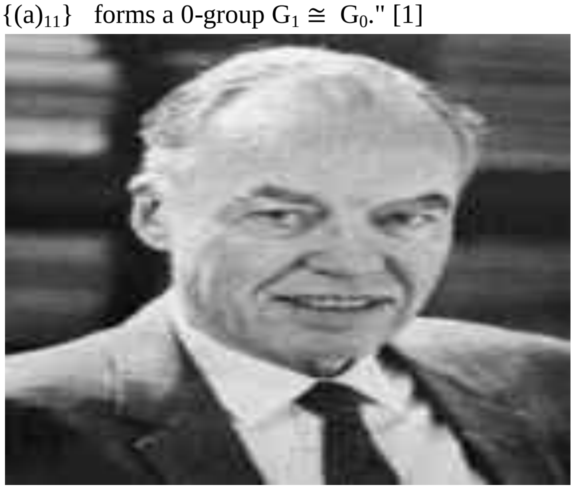
Clifford and Preston worked on minimal ideals and minimal conditions. They together proved "let $S=S^0$ and R be a 0-minimal right ideal of S.Then either rS=R for every r in R\0 or else R={0,r} with rS=0. Also described an Ideal generated 0-minimal right ideals and worked on the combined theory of left and right socles[4,5].

Alfred Clifford studied the union of groups in 1933, and he published many papers. Many researchers adopted Clifford's results and methods for their work. "The Algebraic Theory of Semigroups" by Clifford and Preston was published first. His book "Modern theory of Semigroup" gave new ideas to many researchers.

Mainly he worked on

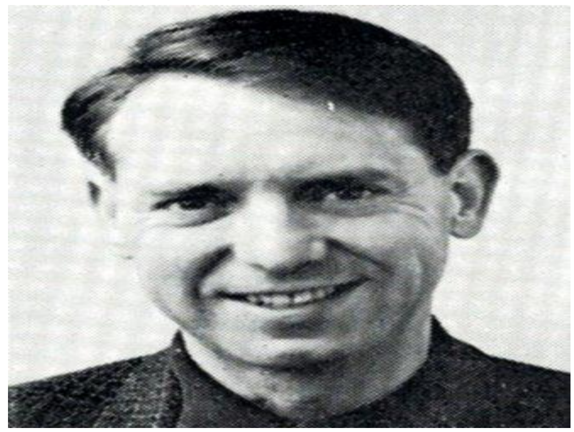
- Rees theorem
- Completely 0-simple
- Principle ideals of Semigroup.
- Semilattice
- Special types of completely regular semigroups.
- The matrix representation theorem is as follows.

"A matrix representation of a semigroup Ş is a morphism $I:Ş\rightarrow M_n(\Omega)$ where $M_n(\Omega)$ denotes the multiplicative Semigroup of nxn matrices with entries from a field $\mathbf{\Omega}$. Take 0-simple Semigroup S, represented as Rees matrix semigroup with elements written in $(a)_{i\lambda}$. Normalize sandwich matrix P in such a way that all entries are either 0 or e, in particular, arranged so that $p_{11}=e$ then $(a)_{11}(b)_{11}=(ab)_{11}$ here $\{(a)_{11}\}$ forms a 0-group $G_1 \cong G_0$." [1]



W.DOUGLAS MUNN (1929-2008):

- In the theory of representation of a finite group G by matrices over a field F, the concept of the algebra of G over F plays a fundamental part. If" F has characteristic zero or prime not dividing the order of G, then this algebra is semisimple. In consequence, the representation of G over F is completely reducible."[1]
- Went on to build on Clifford's work by constructing an irreducible representation of a finite 0-simple semigroup from its structure group
- He has developed Inverse semigroups, bicyclic semigroups, and semilattice.



J. S. PONIZOVSKII (1928-2012):

- Studied P-systems: Semigroups whose semigroup algebras are semisimple.
- Constructed all irreducible from the representation of a Rees matrix semigroup from those of its structure group.
- His published paper is" On Simple subsemigroups of groups."
- A problem "where a simple subsemigroup of a group is a subgroup" is considered. "A simple semigroup means a semigroup S with no ideals different from S."[6]

FUNDING STATEMENT

No funding has been received for this article.

CONCLUSION

In this paper, the reader can easily understand the root of the Semigroup theory and the extraordinary development of Algebraic structure. This paper covered the history of semigroups' founders. This basic content provides the reader with a variety of special cases.

ACKNOWLEDGMENT

"I want to thank my supervisor and author, John M.Howie's textbooks, as I perceived by referring to his journals and books about Semigroups."

REFERENCE

[1] J.M Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.
[2] Hollings, C. (2009). The Early Development of the Algebraic Theory of Semigroups. Archive for History of Exact Sciences, 63, 497-536.
[3] MARK V. LAWSON, LIAM O'CARROLL, AND SARAH REES, "A Tribute to David Rees," HERIOT-WATT UNIVERSITY.
[4] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys No. 7, Amer. Math. Soc., Providence, 1961.
[5] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. II, Mathematical Surveys No. 7, Amer. Math Soc., Providence, 1967.
[6] Ponizovskii, J.S.. "On a type of matrix semigroup.." Semigroup forum 44.1 (1992): 125-128. <http://eudml.org/doc/135136>.
[7] J.M Howie An Introduction to Semigroup theory, academic press, London,, 1976.