Sir William Rowan Hamilton's quarternion: a fundamental insight and new picture of orthogonality

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Abstract **Quarternions are visualized in lesser dimensions. The new idea of orthogonality is proposed. We imagined in this paper the quaternions concepts and quaternions algebra by ultimately basic concepts. Representation of quaternions by coordinates and transformation of quaternion in space, equivalently rotation in terms of displacements.**

Index Terms **orthogonality in different dimensions, rotation as multiplication by complex number a part of quarternion., Quaternion, quarternions as coordinates**

I. INTRODCTION OF OTHOGONALITY

We start with two vectors perpendicular in 2-dimensional space and the simple geometrical figure is

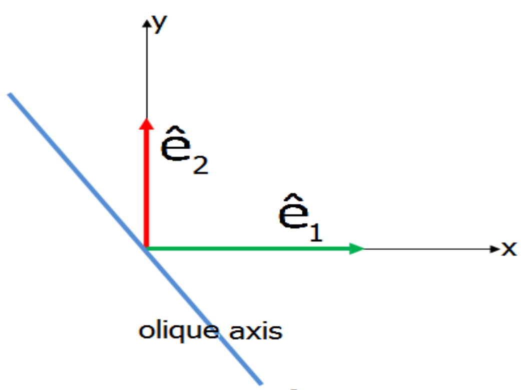


fig.1

As the number of dimensions is equal to the number of vectors to be drawn as orthogonal and there is no difference in perpendicularity and orthogonality. Now we want to see the same perpendicular vectors in one – dimension, one less than the previous one, and the solution is we have to project these two on the oblique axis. On this new axis the projections are |↑> and |↓>. This happens in Stern-Gerlach experiment. We should accept this process of taking projection as fundamental permission of nature. In this process the angle changes from π/2 to π [fig.1]. With this angle of π between two vectors we have to call them orthogonal or perpendicular. In one dimensional situation there will be three points on a line But here nature allows new definition and in my opinion it is entirely unseen in previous literature. Now we define

$$\langle \hat{e}_1 \circ \hat{e}_2 \rangle_1 = |\hat{e}_1||\hat{e}_2| \cos_1^* \pi = [\hat{e}_1 \bullet \hat{e}_2]_{2-\text{dim}} = |\hat{e}_1||\hat{e}_2| \cos\tfrac{\pi}{2} = 0 = |\hat{e}_1||\hat{e}_2| \sin\pi \quad (1)$$

This astonishing definition is the law of nature.

In this picturization in reduced dimension from 3-dimensions [fig.2] to 2-dimensions, one less than the previous one, we see that angle changes from π/2 to 2π/3 [fig.3].

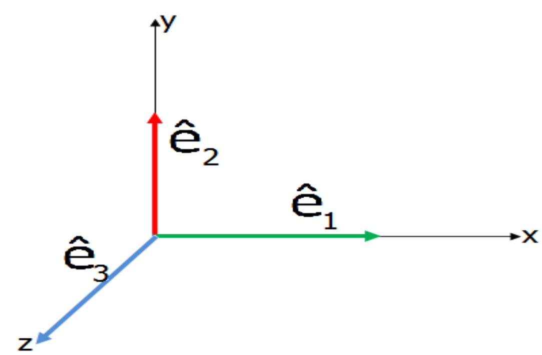


fig. 2

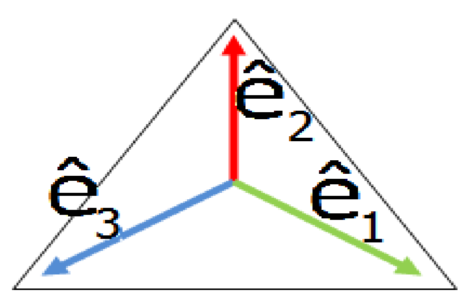


fig. 3

Now we ought to define

$$\langle \hat{e}_i \circ \hat{e}_j \rangle_2 = |\hat{e}_i||\hat{e}_j|\cos^*{}_2 \tfrac{2}{3}\pi = [\hat{e}_i \bullet \hat{e}_j]_{3-\dim} = |\hat{e}_i||\hat{e}_j|\cos\tfrac{1}{2}\pi = 0$$

…(2)

Now we imagine 4 orthogonal vectors in 4-dimensions and in one less than this, i.e., 3-dimensions and we get the methane like structure. In which hydrogen atoms are placed along dimensional axes and carbon atom is at the origin [fig.4].

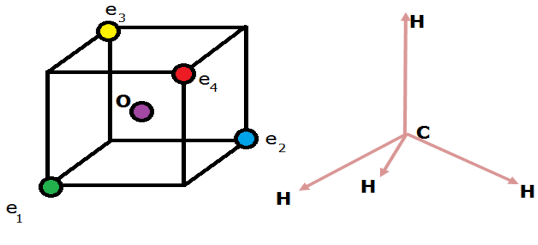


fig.4

$$\langle \hat{e}_i \circ \hat{e}_j \rangle_3 = |\hat{e}_i||\hat{e}_j|\cos^*{}_3(\cos^{-1}(-\tfrac{1}{3})) = [\hat{e}_i \bullet \hat{e}_j]_{4-\dim} = |\hat{e}_i||\hat{e}_j|\cos\tfrac{1}{2}\pi = 0$$

…(3)

In this way we shall picturize the 4 orthogonal vectors in 4- dimensions as well as in reduced imensions. Our finding is we should accept the following reality to picture the orthogonality of n-vectors in (n-1)-dimensions :

$$\langle \hat{e}_i \circ \hat{e}_j \rangle_{n-1} = |\hat{e}_i||\hat{e}_j|\cos^*{}_{n-1} \lambda_n \pi = 0 \quad (4)$$

Where λ_n depends upon geometry and manifests the induction in the system.

II. INTRODCTION OF QUARTERNION

Now we use above consideration to quarternions.

$$i = \sqrt{-1}, \quad j = \sqrt{-1}, \quad k = \sqrt{-1}$$

$$q = a1 + bi + cj + dk, \qquad q^* = a1 - bi - cj - dk$$

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

$$|q|^2 = qq^* = a^2 + (-i^2)b^2 + (-j^2)c^2 + (-k^2)d^2 + [i,j]_+ bc + [j,k]_+ cd + [k,i]_+ bd$$

$$\therefore i^2 = j^2 = k^2 = -1, \quad [i,j]_+ = [j,k]_+ = [k,i]_+ = 0$$

…(5)

$$[a \;,\; b]_{\pm} = ab \pm ba$$

Hence we are, ultimately, left with anti-commutativity.

Further we notice that

$$[ij,i]_+ = 0 = [ij,j]_+,$$
$$[jk,j]_+ = 0 = [jk,k]_+,$$
$$[ki,k]_+ = 0 = [ki,i]_+ \qquad (6)$$
$$[ijk,i]_- = [ijk,j]_- = [ijk,k]_- = 0$$

On close inspection of Eq.(5) and Eq.(6), we observe that ijk may have properties of real numbers while the set {ij,jk,ki}is not totally different from the set {i,j,k}, and we can say

$$ij = c_1 k, \quad jk = c_2 i, \quad ki = c_2 j,$$
$$|c_1| = |c_2| = |c_3| = 1$$

…(7)

If we choose $c_1 = c_2 = c_3 = 1$, we obtain Hamilton's quarternion[4,5,6] :

$$ij = k, \quad jk = i, \quad ki = j,$$
$$ijk = -1 \qquad \text{…(8)}$$

Here what we get is the Hamilton's genius equation from algebraic calculations. For the physical insight of anticommutativity, we may consider finite rotations where commutation ceases[7]. We associate these i, j, k with rotations i.e., these entities acquire the operator characters

III QUARTERNION AS ROTATION

In 2- dimensions, we see that multiplication with i is concerned with rotation of π/2 [fig. 5]. Mathematically, we write,

$$i\;(1,0) = (0,1)$$

…(9)

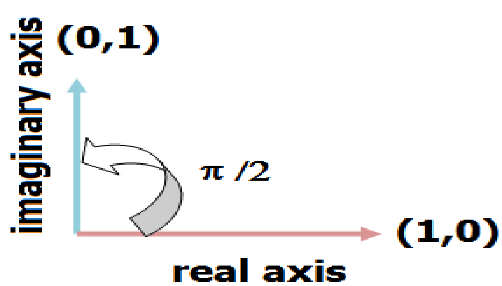


fig. 5

Now I need to implement above concepts for quarternions mapped into 3-dimensions (Methane structure).

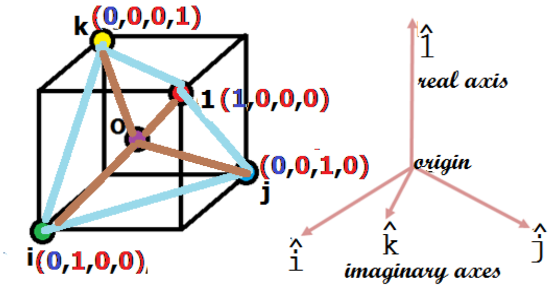


fig. 6

When we deploy co-ordinates of 3-dimensions, we have the familiar representation:

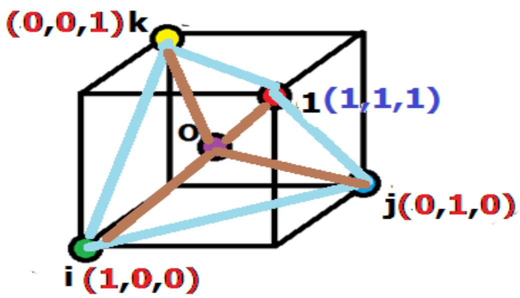


fig.7

If we drop the real axis, there is no need of methane structure and we represent imaginary entities as perpendicular axes:

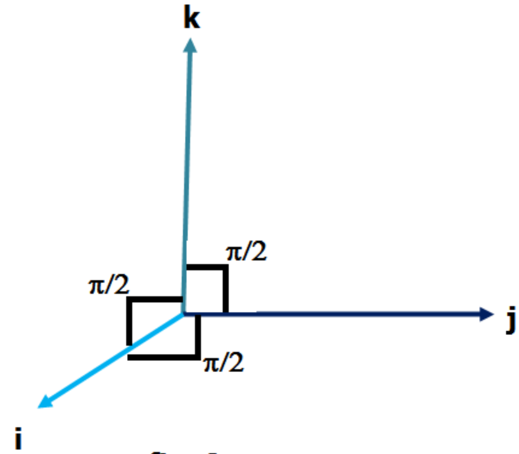


fig. 8

Eq. (9) and fig.5 reveal that multiplication of i is equivalent to rotation of axis (1,0) to axis (0,1) or the translation of point (1,0) to point (0,1).

IV. USE OF CO-ORDINATES FOR ROTATION [1,2,3]

Looking of rotation as translation makes our calculation the easiest ones.

$$i(1,1,1) = (1,0,0) \Leftrightarrow (1,1,1) \ \text{to} \ (1,0,0) \equiv \text{rotation about point} \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$\text{or translation on front face}$$
$$j(1,1,1) = (0,1,0) \Leftrightarrow (1,1,1) \ \text{to} \ (0,1,0) \equiv \text{rotation about point} \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$\text{or translation on right face}$$
$$k(1,1,1) = (0,0,1) \Leftrightarrow (1,1,1) \ \text{to} \ (0,0,1) \equiv \text{rotation about point} \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$\text{or translation on top face} \qquad (10)$$

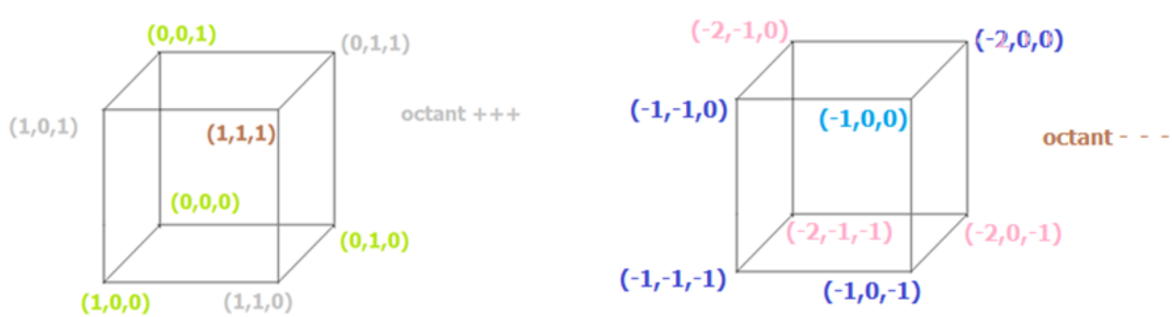


fig. 9

$$i(1,1,1) = (1,0,0) \Leftrightarrow (1,1,1) \quad to \quad (1,0,0) \quad rotation\ about\ point\ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$or\ translation\ on\ front\ face \equiv (1,1,1) + (0,-1,-1) = (1,0,0)$$
$$j(1,1,1) = (0,1,0) \Leftrightarrow (1,1,1) \quad to \quad (0,1,0) \quad rotation\ about\ point\ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$or\ translation\ on\ right\ face \equiv (1,1,1) + (-1,0,-1) = (0,1,0)$$
$$k(1,1,1) = (0,0,1) \Leftrightarrow (1,1,1) \quad to \quad (0,0,1) \quad rotation\ about\ point\ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$or\ translation\ on\ top\ face \equiv (1,1,1) + (-1,-1,0) = (0,0,1)$$
$$k \equiv observer\ is\ at\ point\ (0,0,1)$$
$$jk \equiv (1,1,1) \quad to \quad (0,1,0)\ translation\ after\ point\ (0,0,1)$$
$$= shift\ point\ (1,1,1)\ to\ point\ (0,0,1)\ and\ then\ apply\ translation\ on\ right\ face$$
$$= (1,1,1) + (-1,-1,0)\ and\ then\ apply\ translation\ on\ right\ face$$
$$= (1,1,1) + (-1,-1,0) + (-1,0,-1) = (-1,0,0)$$
$$ijk = (1,1,1) \quad to \quad (1,0,0) \quad rotation\ about\ point\ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\ at\ point\ (-1,0,0)$$
$$= (1,1,1) + (-2,-1,-1)\{shifting\} + (0,-1,-1)\{rotation\} = (-1,-1,-1) \equiv -1 \qquad \ldots(11)$$

In this way ijk, transforms the real quantity 1 (octant + + +) into the real quantity -1
(octant - - -) [fig. 9].

V. QUARTERNION AND VECTORS

It is interesting to imagine that why should we impose restriction of one imaginary part only?

Now we remove this restriction and define new quantity and say it *quaternion*.

$$\| H \|^2 = \{a + xi + yj + zk\}\ \{a - xi - yj - zk\}$$
$$= a^2 - x^2 - y^2 - z^2 - xy(ij + ji) - yz(jk + kj) - zx(ki + ik) \qquad \ldots(17)$$

Equating (3) and (6), we see (5) is automatically satisfied. But one more requirement is

(ij+ji)=0, (jk+kj)=0, (ki+ik)=0 …(18)

Eq (7) says that three quantities i , j , k are *non-commutative*.

<u>Vectors</u>

H=a +ix+jy+zk. …(12)

Then H*=a-ix-jy-zk …(13)

$\|H\|^2=a^2+ x^2+ y^2+ z^2$ …(14)

$\|i\|=\|j\|=\|k\|=1$ …(15)

$i^2= j^2= k^2= -1$ …(16)

$$\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z} \quad ,\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$$

We introduce corresponding quaternion as

$$\tilde{a} = a_1 i + a_2 j + a_3 k \quad ,\ \tilde{b} = b_1 i + b_2 j + b_3 k \qquad \ldots(19)$$

$$\tilde{a}\tilde{b} = (a_1 i + a_2 j + a_3 k)\ (b_1 i + b_2 j + b_3 k)$$

With the help of (5) and (7), we have

$$\tilde{a}\tilde{b} = (a_1 i + a_2 j + a_3 k)\ (b_1 i + b_2 j + b_3 k) = -\vec{a} \bullet \vec{b} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} [ij] + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} [jk] + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} [ki] \qquad \ldots(20)$$

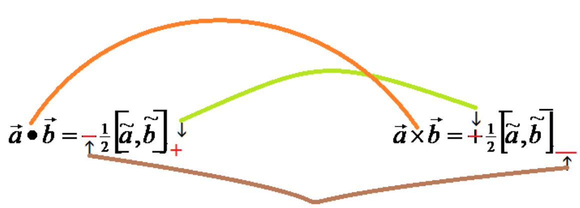
For proper resemblance to known vector product we must assume

ij=k, jk=i, ki=j …(21)

This is usual *cyclic* property.

$$\tilde{a}\tilde{b} = -\vec{a} \bullet \vec{b} + \vec{a} \times \vec{b}$$
$$[a,b]_{\pm} = ab \pm ba$$
$$\vec{a} \bullet \vec{b} = -\tfrac{1}{2}[\tilde{a},\tilde{b}]_{+} \qquad \vec{a} \times \vec{b} = \tfrac{1}{2}[\tilde{a},\tilde{b}]_{-}$$

:



Using (5), (10) can also be written as

i j k = -1 …(22)

Combining (16) and (22),we obtain the combination of (5)and (8),thus the *quaternion* can be *well- known* as

$$i^2 = j^2 = k^2 = ijk = -1 \qquad \ldots(23)$$

VI.CONCLUSION

In this paper we used geometrical methods especially coordinates to probe into the mechanism of understanding Sir Hamilton's quarternions. The reduction of dimensions is more beneficial for the purpose. It is interesting to claim that commutability is lost as the finite rotations do not commute. Differential operators and matrices are well known entities which do not obey commutativity. Quaternions also show this feature and should have a wide variety of applications in modern physics.

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