Estimation of Default Risk using Stochastic Volatility Models: The case of Heston-CIR Model

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Abstract: **The volatility of an asset is a key component to pricing options. Stochastic volatility models were developed out of a need to modify the Black Scholes model for pricing options, which failed to effectively take the fact that the volatility of the price of the underlying security can change into account. The Black Scholes model instead makes the simplifying assumption that the volatility of the underlying security was constant. Stochastic volatility models correct for this by allowing the price volatility of the underlying security to fluctuate as a random variable. By allowing the price to vary, the stochastic volatility models improved the accuracy of calculations and forecasts. This study uses the stochastic volatility model: the Heston-CIR model, which is a combination of the stochastic volatility model discussed in Heston and the stochastic volatility model driven by Cox-Ingersoll-Ross (CIR) processes to predict the default risk and compare the results with the Merton jump diffusion (MJD), the traditional Merton and the Moody's KMV (MKMV) models. Results show that, the Heston-CIR model predicts accurately the default risk as compared to other models.**

Index Terms: **Default Risk, Stochastic volatility model, Heston-CIR model, MJD Merton model, the Merton model, MKMV model.**

I. INTRODUCTION

The variance of a stochastic process is randomly distributed in stochastic volatility models. The concept of stochastic volatility recognizes that asset price volatility is not constant but rather fluctuates over time. Many fundamental options pricing models, notably Black Scholes, use the assumption of constant volatility, which leads to pricing inefficiencies and mistakes. Stochastic volatility modelling, which allows volatility to alter over time, addresses Black Scholes' issue. The pattern of mean reversion causes the stochastic volatility models to always return to a fixed long-run mean if the current level deviates from the mean value [7].

The volatility of an asset is an important consideration when pricing options. Stochastic volatility models were developed to improve the Black Scholes (1973) model for pricing options, which failed to fully account for the likelihood of underlying security price volatility. Instead, the Black Scholes model assumes that the volatility of the underlying security is constant. Stochastic volatility models accommodate for this by allowing the price volatility of the underlying investment to fluctuate randomly. By allowing the price to fluctuate, stochastic volatility models improved the precision of computations and forecasts [18]

In this work, we employ a stochastic volatility model; the Heston-CIR model, to predict default risk in stock markets and compare the findings to the Merton Jump Diffusion (MJD), classic Merton, and Moody's KMV (MKMV) techniques.

II. LITERATURE REVIEW

Merton (1974) proposed a method for calculating a firm's credit risk by considering the asset as a call option. They modelled a firm's asset value as a lognormal process, with the assumption that if the asset value went below a certain default boundary, the firm would fail. At maturity, the default choice was only offered once. The advantage of this model is that it can be used to any publicly traded company, and stock market data can be utilized instead of financial data. It can also be used to forecast events in the future. The use of this approach in ordinary practice, on the other hand, revealed some of its faults. The model's credit spreads, which are premiums to risk-free interest rates, are often lower than the real spreads, and Merton's model's assumptions have little resemblance to reality. The model's credit spreads, which are premiums to risk-free interest rates, are often lower than the real spreads, and Merton's model's assumptions have little resemblance to reality [12].

Merton (1976) created a jump stochastic process that incorporates jumps (non-local changes) in a continuous random time. No matter how short the time delay between successive observations, the approach provided for a positive likelihood of a stock price change of extraordinary magnitude. Many empirical investigations of stock price series show much too many outliers for a basic constant-variance lognormal distribution, showing the presence of price leaps [13].

Sepp (2006) provided two robust CreditGrades model extensions. The first extension assumes that the variance of returns on the firm's assets is stochastic, and the second that the asset value process of the firm follows a double exponential jump-diffusion. They developed closed-form methods for pricing equity options on a firm and calculating the firm's survival probability over a finite time horizon. Their models were used to simulate credit default swap (CDS) and equity default swap (EDS) spreads. They calibrated their models using data from General Motors choices. Their models offered a good fit to the data and resulted in non-zero short term CDS spreads, according to the results [1].

Jacobs and Li (2008) studied a two-factor affine model for corporate bond credit spreads. The first factor was understood as the spread's level, and the second as the spread's volatility. A basic two-factor affine model was used to simulate the riskless interest rate, yielding a four-factor model for corporate yields. They were able to capture higher moments of credit spreads by modelling the volatility of corporate credit spreads as stochastic. They estimated their model on corporate bond prices for 108 corporations using an enhanced Kalman filter method. Their model fit actual corporate bond credit spreads well, resulting in a much lower root mean square error (RMSE) than a typical alternative model in both in-sample and out-of-sample tests. The model also caught the essential characteristics of real-world credit spreads [10].

Masoliver and Perelló (2009) solved the Heston random diffusion model's first-passage difficulty. They were able to obtain precise analytical formulas for the survival and hitting probabilities at a given level of return. They investigated numerous asymptotic behaviours and obtained approximate representations of these probabilities, demonstrating, among other things, the nonexistence of a mean-first-passage time. One important finding was the presence of extreme deviations, implying a high chance of default when a dimensionless parameter related to the strength of volatility fluctuations increased. They tested the model on empirical daily data and discovered that it could capture a relatively broad region of hitting probability [9].

Gersbach and Surulescu (2010) devised a method for estimating default risk based on stochastic volatility models. Instead of Merton's standard technique with geometric Brownian movements, they considered a mean-reverting stochastic volatility model to represent the evolution of a firm's values. For the default probability, they devised an analytical equation. Their simulation results showed that if a firm's credit quality is not excessively poor, the stochastic volatility model predicts higher default probabilities than the related Merton model. Otherwise, the stochastic volatility model predicts lower default probabilities [8].

Markovska et al. (2014) used the Cox-Ingersoll-Ross model to estimate defaults using an intensity-based method. They investigated the potential and consequences of the theoretical model's non-linear dependency between economic and financial state variables and the default density. They then ran a test to see how simulation techniques can assist the study of such complex relationships when closed-form solutions are either unavailable or difficult to get. Their technique enabled them to create basic and easy-to-implement models for assessing default risk [16].

Hurn and Lindsay (2015) presented a maximum likelihood method for estimating the parameters of the classic square-root stochastic volatility model as well as a variant of the model that includes equity price jumps. They applied the model on S&P 500 Index data and vanilla option prices written on the index from 1990 to 2011. Their method was capable of estimating both the physical measure parameters (related with the index) and the risk-neutral measure parameters (associated with the options), including the volatility and jump risk premia. The empirical results showed that the model parameters could be successfully determined and were compatible with values reported in the literature. Both the volatility risk premium and the jump risk premium were discovered to be statistically significant [2].

Chen and He (2017) investigated the pricing of credit default swaps (CDSs) using a reference asset driven by a geometric Brownian motion with multi-scale stochastic volatility (SV), which was a two-factor volatility process with one factor controlling the fast time scale and the other representing the slow time scale. A crucial component was the discussion of no default probability, which established an equivalence link between the CDS and the down-and-out binary option while balancing the two SV processes with the perturbation method. Finally, they were able to develop an approximate but closed-form pricing formula for the CDS contract [17].

Wang et al. (2017) investigated the pricing issue of stochastic volatility in susceptible options by decomposing stochastic volatility into long-term and short-term volatility. They used a mean-reverting process to represent the short-term variation of stochastic volatility and assumed long-term volatility to be constant. They developed a pricing formula for susceptible options in a special situation using the proposed model. They compared the suggested model's results to those of Black and Scholes (1973), Heston (1993), and Klein (1996). The stochastic volatility model was discovered to be a flexible representation of susceptible option pricing when the parameters were properly chosen [7].

McQuade (2018) created a firm-specific real-options, term structure model to provide fresh insight on the value premium, financial crisis, momentum, and credit spread challenges. The model included stochastic volatility in the company productivity process as well as a negative volatility risk market price. The model demonstrated that allowing for endogenous default by equity holders was required for the model to account for both investment grade and junk debt credit spreads. To better account for the projected default frequencies and credit spreads of short maturity debt, the model was expanded to include infrequent disasters and numerous time scales in volatility dynamics. Finally, they solved the model using an asymptotic expansions-based methodology [15].

Ji et al. (2020) modified the Merton model by including stochastic volatility and the concept of undercapitalization to more realistically assess bank credit risk. They chose the Heston model, which has non-lognormal features such as strong tails in the asset return distribution. To estimate parameters, they used Bayesian inference. Then, to better show banks' credit risk, they created capital adequacy requirements, and they offered an early warning indication, namely the ECB. They used the ECB to bail out Lehman Brothers and Bank of America. The results indicated the relative power of their early warning indication vs the put option value of the bank's safety net [5].

Jumbe and Gor (2022) devised a method for modelling default risk using the jump diffusion process. They compared the outcome of their method to that of Merton and Moody's famous Kealhofer, McQuown, and Vasicek (MKMV) models. The results show that jump diffusion models outperform both the standard Merton and MKMV models in predicting default risk [6].

III. STOCHASTIC VOLATILITY MODELS

The following is a general expression for a non-dividend asset price with stochastic volatility:

$$dA_t = \mu_t A_t dt + \sqrt{v_t} A_t dW_{1t} \quad (1)$$

$$dv_t = \alpha\left(S_t, v_t, t\right)dt + \beta\left(A_t, v_t, t\right)dW_{2t} \quad (2)$$

with

$$dW_{1t} dW_{2t} = \rho dt \quad (3)$$

where A_t denotes the asset price and v_t denotes the variance of the asset price.

IV. THE COX-INGERSOLL-ROSS (CIR) MODEL

John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross developed the Cox-Ingersoll-Ross (CIR) model in 1985 as an expansion of the Vasicek model (1984). The CIR model was originally used to represent the evolution of interest rates. It is a one factor model (short-rate model) since interest movements are described as being driven by only one source of market risk. The concept is now extended to characterize volatility evolution in stochastic volatility models.

The CIR model states that the instantaneous volatility follows the stochastic differential equation known as the CIR process, which is given as:

$$dv_t = a\left(b - v_t\right)dt + \sigma\sqrt{v_t}\, dW_t \quad (4)$$

where W_t is a wiener process (modelling the random market risk component), a is the speed of adjustment to the mean, b and σ is the variance volatility. The drift factor, $a\left(b - v_t\right)$, is identical to that of the Vasicek model. It guarantees mean reversion of the volatility rate to the long run value b, with the speed of adjustment set by a strictly positive parameter, a.

For any positive values of a and b, the standard deviation factor, $\sigma\sqrt{v_t}$, eliminates the possibility of negative volatility rates. If the condition $2ab \geq \sigma^2$ is met, a volatility rate of zero is likewise ruled out. When the rate v_t approaches zero, the standard deviation $\sigma\sqrt{v_t}$ likewise approaches zero, dampening the influence of the random shock on the rate. When the rate approaches zero, its evolution is driven by the drift factor, which forces the rate towards equilibrium.

V. THE HESTON MODEL

The Heston Model is a form of stochastic volatility model used to price European options, named after Steve Heston (1993). The Heston methodology is a stochastic volatility options pricing methodology. In contrast to the Black-Scholes model, which assumes that volatility is constant, the model assumes that volatility is arbitrary. It is a closed-form method for option pricing that attempts to address some of the drawbacks of the Black-Scholes option pricing model. It is also a type of volatility smile model, which is a graphical representation of numerous options with identical expiration dates that demonstrate increased volatility as the options become more in-the-money (ITM) or out-of-the-money (OTM).
The Heston Model employs statistical methods to calculate and forecast option pricing under the assumption that volatility is arbitrary. The premise that volatility is arbitrary rather than constant is what distinguishes stochastic volatility models. The SABR model, the Chen model, and the GARCH model are examples of stochastic volatility models.

The basic Heston model assumes that, the asset price A_t is determined by a stochastic process given by:

$$dA_t = \mu A_t dt + \sqrt{v_t} A_t dW_{1t}, \qquad (5)$$

where v_t is the instantaneous variance given by a CIR process given by:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2t}, \qquad (6)$$

and W_{1t}, W_{2t} are independent wiener processes with correlation ρ.

The model has five parameters, v_0 is the initial variance, θ is the long-run average variance of the price; as t tends to infinity, the expected value of v_t tends to θ, ρ is the correlation of the two wiener processes, κ is the rate at which v_t reverts to θ and σ is the volatility of the volatility which determines the variance of v_t. If the parameters obey the Feller condition $2\kappa\theta > \sigma^2$, then the process v_t is strictly positive.

VI. THE HESTON-CIR MODEL

The Heston-CIR model is a typical stochastic volatility model which takes
$\alpha(A_t, v_t, t) = \kappa(\theta - v_t)$,
and $\beta(A_t, v_t, t) = \sigma\sqrt{v_t}$, to obtain:

$$dA_t = \mu_t A_t dt + \sqrt{v_t} A_t dW_{1t} \qquad (7)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2t} \qquad (8)$$

with

$$dW_{1t}dW_{2t} = \rho dt \qquad (9)$$

where W_{1t} and W_{2t} are Wiener processes for asset price and asset's price variance respectively with instantaneous correlation ρ, A_t is the asset price at time t, r is the risk free interest rate, v_t is the variance of the asset price, σ is the volatility of the volatility (the volatility of the variance), θ is the long-term average price variance (mean reversion level), κ is the mean reversion speed (rate of reversion to θ), the instantaneous variance v_t is a CIR process (square root process) and dt is the indefinitely small positive time increment. The parameters of the variance process are all strictly positive.
The risk-neutral probability measure incorporates the market price of volatility, denoted as λ, to distinguish the objective probability measure from the risk-neutral one. The volatility risk premium is assumed to be proportional to the instantaneous variance, λv_t, and its sign arises from the (sign of) correlation between the Brownian processes assumed for the instantaneous variance and the (aggregate) consumption.

If we let $x_t = \ln A_t$, the risk-neutral dynamics of Heston-CIR model becomes:

$$dx_t = \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\,dW_{1t}^* \qquad (10)$$

$$dv_t = \kappa^*\left(\theta^* - v_t\right)dt + \sigma\sqrt{v_t}\,dW_{2t}^* \qquad (11)$$

with

$$dW_{1t}^* dW_{2t}^* = \rho dt \qquad (12)$$

where

$$\kappa^* = \kappa + \lambda \text{ and } \theta^* = \frac{\kappa\theta}{\kappa + \lambda}$$

The probability of the call option expiring in-the-money, conditional on the log of the asset price, can be interpreted as risk-adjusted or risk-neutral probabilities provided by:

$$F_j\left(x, v, T; \ln D\right) = P\left(x(T) \geq \ln D \mid x_t = x, v_t = v\right) \qquad (13)$$

The Heston model treats the option price V_t as a function of current asset price A_t and its volatility v (level of volatility), time to expiration T, strike price D, risk-free rate r, variance parameters of asset price and its volatility of underlying asset σ (volatility of variance), mean reversion speed κ, mean reversion level for the variance θ, volatility risk premium λ and correlation between two process ρ.

$$V = V\left(v_t, A_t, T, D, r; \sigma, \kappa, \theta, \lambda, \rho\right) \qquad (14)$$

Heston demonstrates that the value $V\left(A, v, t\right)$ of any option must meet the following partial differential equation using classic arbitrage arguments:

$$\frac{1}{2}vA^2\frac{\partial^2 V}{\partial A^2} + \rho\sigma v A\frac{\partial^2 V}{\partial A \partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2} + rA\frac{\partial V}{\partial A} + \begin{pmatrix}\kappa\left(\theta - v(t)\right) - \\ \lambda\left(A, v, t\right)\end{pmatrix}\frac{\partial V}{\partial v} - rV + \frac{\partial V}{\partial t} = 0 \qquad (15)$$

Heston obtained the closed form solution of a European call option on a non-dividend paying asset using the analogy of the Black-Scholes formula as follows:

$$C\left(D\right) = SP_1 - e^{-r(T-t)}DP_2 \qquad (16)$$

where P_1 and P_2 should satisfy the following Partial Differential Equation (PDE) (for $j = 1, 2$)

$$\frac{1}{2}v\frac{\partial^2 F_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 F_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} + \left(r + u_j v\right)\frac{\partial P_j}{\partial x} + \left(a_j - b_j v\right)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0 \qquad (17)$$

and

$$P_j = \Pr\left(\ln\left(A_t\right) > \ln\left(D\right)\right) \qquad (18)$$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K} f_{j(\phi; x, v)}}{i\phi}\right]d\phi \qquad (19)$$

$$f_j\left(\phi; x, v\right) = \exp\left(C_j\left(\tau, \phi\right) + D_j\left(\tau, \phi\right)v_t + i\phi x_t\right) \qquad (20)$$

$$C_j\left(\tau, \phi\right) = ri\phi\tau + \frac{a}{\sigma^2}\left[\begin{matrix}\left(b_j - \rho\sigma i\phi + d_j\right)\tau - \\ 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\end{matrix}\right] \qquad (21)$$

$$D_j\left(\tau, \phi\right) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \qquad (22)$$

$$d_j = \sqrt{\left(\rho\sigma i\phi - b_j\right)^2 - \sigma^2\left(2u_j i\phi - \phi^2\right)} \qquad (23)$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} \qquad (24)$$

The parameters in Equation (14) are given as;

$$u_1 = \frac{1}{2},\ u_2 = \frac{1}{2},\ a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda \tag{25}$$

The simulated variance can be inspected to check whether it is negative ($v < 0$). If it is negative, it can be set to zero ($v = 0$), or invert its sign to be positive ($-v < 0$).

The variance process can be described similarly to the asset price by establishing a process for natural log variances using Ito's lemma, as follows:

$$d \ln v_t = \frac{1}{v_t}\left(\kappa^*\left(\theta^* - v_t\right) - \frac{1}{2}\sigma^2\right)dt + \sigma\frac{1}{\sqrt{v_t}}dW_{2t}^* \tag{26}$$

The Heston model can be discretized as follows;

$$\ln A_{t+\Delta t} = \ln A_t + \left(r - \frac{1}{2}v_t\right)\Delta t + \sqrt{v_t}\sqrt{\Delta t}\varepsilon_{A,t+1} \tag{27}$$

$$\ln v_{t+\Delta t} = \ln v_t + \frac{1}{v_t}\left(\kappa^*\left(\theta^* - v_t\right) - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\frac{1}{\sqrt{v_t}}\sqrt{\Delta t}\varepsilon_{v,t+1} \tag{28}$$

Shocks to the volatility, $\varepsilon_{v,t+1}$, are correlated with the shocks to the stock price process, $\varepsilon_{A,t+1}$. This correlation is denoted as ρ, so that $\rho = Corr\left(\varepsilon_{A,t+1}, \varepsilon_{v,t+1}\right)$ and the relationship between the shocks can be written as;

$$\varepsilon_{v,t+1} = \rho\varepsilon_{A,t+1} + \sqrt{1-\rho^2}\varepsilon_{t+1} \tag{29}$$

where ε_{t+1} are independently with $\varepsilon_{A,t+1}$

VII. HESTON-CIR MODEL PARAMETER ESTIMATIOM

The Heston-CIR model, $V = f\left(A_t, T, D, r, \sigma, v, \kappa, \theta, \lambda, \rho\right)$ is a function of the underlying asset value, the period to expiration, the strike price, the risk-free rate, and the other six model parameters. Bakshi et al. (1997) proposed a loss function technique for estimating the six model parameters. The loss function approach is a numerical estimating method that finds parameter values that minimize the difference between market and Heston-CIR prices [4]

$$error\left(\sigma, v, \kappa, \theta, \lambda, \rho\right) = V_{mkt} - V_{heston}\left(A_t, T, D, r; \sigma, v, \kappa, \theta, \lambda, \rho\right) \tag{30}$$

$$SSE = \min \sum_{n=1}^{N}\left|error\left(\sigma, v, \kappa, \theta, \lambda, \rho\right)\right|^2 \tag{31}$$

The objective function in equation (31) will be minimized for the cross-sectional sum of square error on each day. The numerical approach is used to calculate values of $\sigma, v, \kappa, \theta, \lambda, \rho$.

The Least Square Method

In the Heston-CIR model, the least squares function for parameter estimation entails minimising the sum of the squared differences between the observed data and the model predictions for each observation. When the errors in the observations are believed to be regularly distributed, this strategy is often used.

The objective function for the least squares estimation of the Heston-CIR model parameters is defined as:

$$LS\left(\theta\right) = \sum_{i=1}^{n}\left(y_i - y\left(\theta, x_i\right)\right)^2 \tag{32}$$

where θ is the vector of model parameters, y_i is the observed value of the dependent variable at time i, x_i is the observed value of the independent variable at time i, and $y\left(\theta, x_i\right)$ is the predicted value of the dependent variable at time i, given the model parameters θ and the independent variable value x_i.

For the Heston-CIR model, the dependent variable is the asset price, and the independent variable is the time. The predicted asset price is obtained by simulating the Heston-CIR model with the given parameter values.

Numerical optimization procedures can be used to minimize the objective function. The parameter values are iteratively adjusted by the algorithm to minimize the sum of the squared discrepancies between the observed data and the model predictions. The least squares estimate method is based on the assumption that the observations' errors are normally distributed and that the model predictions are unbiased.

The Maximum likelihood Function

In the Heston-CIR model, the maximum likelihood function for parameter estimation is the joint probability density function of the observed data given the model parameters. It is calculated as the product of the asset price's probability density function and the volatility process at each observation.

The probability density function of the Heston-CIR model for the stock price process is given by:

$$P\left(A_t \mid A_0, v_0, r, \theta, \kappa, \sigma, \rho, \mathrm{t}\right) = \left(1/\,sqrt\left(2\times pi\times v_t\right)\right)\times \exp\left(-\begin{pmatrix}\ln\left(A_t / A_0\right)- \\ \left(r-v_t/2\right)\times t\end{pmatrix}^2 / \left(2\times v_t\times t\right)\right) \quad (33)$$

where A_t is the asset price at time t, A_0 is the initial asset price, v_0 is the initial variance, r is the risk-free rate, theta is the long-term mean level of variance, kappa is the mean reversion speed of variance, sigma is the volatility of variance, rho is the correlation between the asset price and the variance processes, and t is the time interval.

The probability density function of the Heston-CIR model for the variance process is given by:

$$G\left(v_t \mid v_0, \theta, \kappa, \sigma, \rho, \mathrm{t}\right) = \left(1/\left(\sigma\times sqrt\left(2\times pi\times t\right)\right)\right)\times \exp\left(\begin{matrix}-\begin{pmatrix}v_t-\theta-\left(v_0-\theta\right)\times \\ \exp\left(-\kappa\times t\right)\end{pmatrix}^2 / \\ \left(2\times\sigma^2\left(1-\exp\left(-2\times\kappa\times t\right)\right)\right)\end{matrix}\right) \quad (34)$$

where v_t is the variance at time t, v_0 is the initial variance, theta is the long-term mean level of variance, kappa is the mean reversion speed of variance, sigma is the volatility of variance, and t is the time interval.

The joint probability density function of the Heston-CIR model for the asset price and variance processes is the product of the two probability density functions:

$$L\left(A_t, v_t \mid A_0, v_0, r, \theta, \kappa, \sigma, \rho, \mathrm{t}\right) = P\left(A_t \mid A_0, v_0, r, \theta, \kappa, \sigma, \rho, \mathrm{t}\right)\times G\left(v_t \mid v_0, \theta, \kappa, \sigma, \mathrm{t}\right) \quad (35)$$

The likelihood function is the product of the joint probability density function evaluated at each observation:

$$L\left(\text{Observed data}|A_0, v_0, r, \theta, \kappa, \sigma, \rho, \mathrm{t}\right) = \prod_{i=1}^{n} L\left(A_i, v_i \mid A_0, v_0, r, \theta, \kappa, \sigma, \rho, \mathrm{t}\right) \quad (36)$$

where A_i and v_i are the observed asset price and variance at time i, and n is the total number of observations.

The maximum likelihood estimates of the parameters are the values that maximize the likelihood function. This is typically done using numerical optimization algorithms.

VIII. PROBABILITY OF DEFAULT BY HESTON-CIR MODEL

The Heston-CIR model includes a default threshold (D) and a default indicator function I that equals 1 if its argument is true and 0 otherwise. The default threshold represents the level at which the firm defaults, when the asset value goes below the default threshold and the default indicator function determines if the asset price has fallen below the default threshold.

The probability of default (PD) at time , t is given by:

$$PD(t) = P\left(A_t < D \mid F_t\right) = 1 - N\left(\frac{\ln\left(A/D\right)+\left(r-\frac{1}{2}\sigma^2\right)t}{\sqrt{t}\sqrt{v}}\right) \quad (37)$$

where F_t is the information set at time t, $\sqrt{v}$ is the instantaneous variance, and $N(.)$ is the cumulative distribution function of the standard normal distribution, and; $DD = \dfrac{\ln\left(A/D\right)+\left(r-\frac{1}{2}\sigma^2\right)t}{\sqrt{t}\sqrt{v}}$

(38)
is the distances to default, defined as the number of standard deviations between the expected asset value at maturity T and the debt threshold D, reflecting how far a firm's asset value is from the value of obligations that would trigger a default [19].

IX. DATA ANALYSIS AND DISCUSSION

The data on asset price and default threshold were obtained from Federal Reserve Economic Data,(https://fred.stlouisfed.org,https://fredhelp.stlouisfed.org) from 2011/10/01 to 2020/10/01 as shown in Table 1. The data include total asset prices (A) and total debts (TD) treated as default threshold, short term debts (STD) and long term debts (LTD). Analysis of data was done using python Jupiter IDE.

Distances to default (DDs) determined by the Heston-CIR (HCIR) model, Merton Jump Diffusion (MJD) model, classic Merton model, and Moody's KMV (MKMV) model are shown in Table 2. The table also displays the HCIR model's changing volatility. The table depicts the rise in DDs for the HCIR model when volatility and maturity time rise. As maturity time grows, the DDs for MJD, the classic Merton model, and the MKMV models drop. The HCIR model produces higher DD values than the MJD and classic Merton approaches, but slightly lower values than the MKMV approach. This demonstrates that the HCIR model may be a better way for estimating default risk. Table 3 displays the default probabilities (PDs) computed by the Heston-CIR (HCIR) model, Merton Jump Diffusion (MJD) model, classic Merton model, and MKMV model from Table 1. The table shows how the PDs for the HCIR model fall as volatility and maturity time increase. As maturation time passes, the PDs for MJD, the classic Merton model, and the MKMV model grow. When compared to other techniques, the HCIR model produces lower PD values. This demonstrates that, when compared to other techniques, the HCIR model can be a better predictor of default risk.

X. CONCLUSION AND SUGGESTION FOR FUTURE RESEARCH

We compared stochastic volatility models versus non-stochastic volatility models in predicting default risk in this article. For stochastic volatility models, we utilized the HCIR model, and for non-stochastic volatility models, we used the MJD, conventional Merton, and MKMV models. We used data from the Federal Reserve Economic System from 2011/10/01 to 2020/10/01. We began by calculating and comparing the distances to default (DDs) for all four approaches. We then utilized the DDs to determine the PDs for each approach. The results show that stochastic volatility models give preferable results when compared to non-stochastic volatility models. The DDs produced by the HCIR model were larger than those produced by the MJD and Merton models, but slightly smaller than those produced by the MKMV model. However, as compared to the MJD, Merton, and MKMV techniques, the HCIR model showed lower default probabilities. This demonstrates that the HCIR can be a superior technique for predicting default risk, implying that stochastic volatility models are a better approach for predicting default risk. In the future, we will analyze the effect of changing interest rates on predicting default risk using stochastic interest models and non-stochastic interest models.

REFERENCE

[1] A. Sepp. Extended CreditGrades model with stochastic volatility and jumps. *Wilmott Magazine*, pp.50-62. Sep 2006

[2] A.S. Hurn, K.A. Lindsay and A.J. McClelland, A. Estimating the parameters of stochastic volatility models using option price data. *Journal of Business & Economic Statistics*, *33*(4), pp.579-594. Oct 2015

[3] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, *81*(3), pp.637-654. May 1973

[4] G. Baksh, C. Cao and Z. Chen, Z. Empirical, performance of alternative option pricing, models. *The Journal of finance*, *52*(5), pp.2003-2049. 1997

[5] G. Ji., H. Kong, W.C. Kim and K. Ahn.Stochastic volatility and early warning indicator. In *Computational Science–ICCS 2020: 20th International Conference, Amsterdam, The Netherlands, June 3–5, 2020, Proceedings, Part I 20* (pp. 413-421). Springer International Publishing. 2020

[6] G. Jumbe and R. Gor. Modeling of Default Risk by Jump Diffusion Process. *IOSR Journal of Mathematics (IOSR-JM)*, 18(3): pp. 41-47. 2022

[7] G. Wang, X. Wang and K. Zhou. Pricing

vulnerable options with stochastic volatility. *Physica A: Statistical Mechanics and its Applications*, *485*, pp.91-103. Nov 2017

[8] H. Gersbach and N. Surulescu. Default risk in stochastic volatility models. *CER-ETH (Center of Economic Research at ETH Zurich) Working Paper*, (10/131). Jun 2010

[9] J. Masoliver and J. Perelló. First-passage and risk evaluation under stochastic volatility. *Physical Review E*, *80*(1), p.016108. Jul 2009

[10] K. Jacobs and X. Li. Modeling the dynamics of credit spreads with stochastic volatility. *Management Science*, *54*(6), pp.1176-1188. Jun 2008

[11] P. Klein. Pricing Black-Scholes options with correlated credit risk. *Journal of Banking & Finance*, *20*(7), pp.1211-1229.1996

[12] R.C. Merton,-On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance*, *29*(2), pp.449-470. May 1974.

[13] R.C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, *3*(1-2), pp.125-144. Jan 1976

[14] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, *6*(2), pp.327-343. Apr 1993

[15] T. McQuade. Stochastic volatility and asset pricing puzzles. *Available at SSRN 3222902*.2018

[16] V. Markovska, S. Kabaivanov and M. Milev. Estimation of Default Risk in CIR++ model simulation. *Int. J. Eng*, *1*(1), pp.1-8. 2014

[17] W. Chen and X. He. Pricing credit default swaps under a multi-scale stochastic volatility model. *Physica A: Statistical Mechanics and its Applications*, *468*, pp.425-433. Feb 2017

[18] W. Wattanatorn. Beyond Black-Scholes: the stochastic volatility option pricing model and empirical evidence from Thailand. In *Proceedings Conference of Professor Sangvien Indaravijiya on Thai Financial Market, on* (Vol. 13). Nov 2014

[19] X. Chen, X. Wang and D.D.Wu,-Credit risk measurement and early warning of SMEs: An empirical study of listed SMEs in China. *Decision support systems*, *49*(3), pp.301-310. Jun 2010.

Table 1. Short and long term debts, total debts and total asset prices

Time (T)	2011/10/01	2012/10/01	2013/10/01	2014/10/01	2015/10/01	2016/10/01	2017/10/01	2018/10/01	2019/10/01	2020/10/01
STD	3810	3829	3813	4177	5900	4336	3705	3585	4775	6003
LTD	16487	16947	19431	22299	30692	32037	29130	29690	28792	29921
Asset(A)	173063	171211	191450	205093	203037	198507	201953	211339	228884	253764
Debts (D)	20297	20776	23244	26476	36592	36373	32835	33275	33567	35924

Source (Federal Reserve Economic Data

Table 2. Distances to default (DDs) for HCIR, MJD and Merton models

Time (T)	1	2	3	4	5	6	7	8	9	10
DD(HCIR)	3.4200	4.0777	4.5244	4.8744	5.1673	5.4222	5.6497	5.8565	6.0469	6.2241
Volatility(HCIR)	0.3162	0.4472	0.5477	0.6325	0.7071	0.7746	0.8366	0.8944	0.9487	1.0
DD(MJD)	6.7108	4.9692	4.2410	3.8339	3.5754	3.3989	3.2727	3.1793	3.1086	3.0543
DD(MTN)	6.4108	4.5448	3.7205	3.2304	2.8968	2.6512	2.4608	2.3078	2.1814	2.0747
DD(MKMV)	8.2647	5.8559	4.7909	4.1574	3.7259	3.4081	3.1616	2.9633	2.7993	2.6610

Table 3. Probability of default (PDs) for HCIR, MJD, and Merton models

Time (T)	1	2	3	4	5	6	7	8	9	10
PD (HCIR)	0.0003	2.3e-05	3.0e-06	5.5e-07	1.2e-07	2.9e-08	8.0e-09	2.4e-09	7.4e-10	2.4e-10
PD (MJD)	9.7e-12	3.4e-07	1.1e-05	6.3e-05	0.0002	0.0003	0.0005	0.0007	0.0009	0.0011
PD(MTN)	7.2e-11	2.7e-06	9.9e-05	0.0006	0.0019	0.0040	0.0069	0.0105	0.0146	0.0190
PD(MKMV)	1.1e-16	2.4e-09	8.3e-07	1.6e-05	9.7e-05	0.0003	0.0008	0.0015	0.0026	0.0039