Regular Restrained Domination in Middle Graph

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Abstract: In this paper, we introduce the new concept called regular restrained domination in middle graph. A set S ⊆ V[M(G)] is a restrained dominating set if every vertex in V-S is adjacent to a vertex in S and another vertex in V-S. Note that every graph has a restrained dominating set, since S=V is such a set. Let γ_{rr}[M(G)] denote the size of a smallest restrained dominating set. Also we study the graph theoretic properties of γ_{rr}[M(G)] and many bounds were obtained in terms of elements of G and its relationships with other domination parameters were found.

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INTRODUCTION

In this paper, we follow the notations of [4]. All graphs considered here are simple and finite. As usual p= |V| and q= |E| denote the number of vertices and edges of a graph respectively.

In general, we use <X> to denote the subgraph induced by the set of vertices X and N(v) (N[v]) denote the open (closed) neighbourhoods of a vertex v.

The notation $\propto_0(G)(\propto_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. The notation $\beta_0(G)(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of a vertex (edge) of G. Let deg (v) is the degree of a vertex v and as usual $\delta(G)(\Delta(G))$ is the minimum (maximum) degree.

A middle graph M(G) of a graph G is the graph in which the vertex set is V(G)UE(G) and two vertices are adjacent if and only if either they are adjacent edges of G or one is vertex of G and the other is an edge incident with it.

We begin by calling some standard definitions from domination theory.

A set S⊆V(G) is said to be a dominating set of G, if every vertex in V-S is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by γ(G)[6].

A dominating set S is called the total dominating set, if for every vertex v∈V, there exists a vertex u∈S, u≠v such that u is adjacent to v. The total domination number of G, denoted by γ_t is the minimum cardinality of a total dominating set of G. This is due to E.J.Cockayne, R.M.Dawes and S.T.Hedetniemi [1].

In [7], a connected dominating set D to be a dominating set D whose induced subgraph <D> is connected. The connected domination number γ_c(G) of a connected graph G is the minimum cardinality of a connected dominating set.

A dominating set D of a graph G=(V,E) is a split dominating set if the induced subgraph <(V-D)> is disconnected. The split domination number γ_s(G) of a graph G is the minimum cardinality of a split dominating set developed by Kulli [8].

A dominating set D of a graph G is a cototal dominating set if the induced subgraph <V-D> has no isolated vertices. The cototal domination number γ_{cot}(G) of a graph G is the minimum cardinality of a cototal dominating set. See [8].

In this paper, we study the graph theoretic properties of γ_{rt}[M(G)] and many bounds were obtained in terms of elements of G. Also relationships with other domination parameters were found.

The concept of Roman domination was introduced by, E. J. Cockayne, E.J. Dreyer Jr, S.M. Hedetniemi in [2].

A Roman dominating function on a graph G(V,E) is a function f: V→{0,1,2} satisfying the condition that every vertex u for which f(u)=0 is adjacent to at least one vertex v for which f(v)=2. The weight of a Roman dominating function is the value

$$f(V) = \sum_{u \in V} f(u)$$

The minimum weight of a Roman dominating function on a graph
G is called the Roman domination number of G.
In [3], defined the restrained domination number such that a dominating set D is said to be a restrained dominating set if every vertex of V-D is adjacent to a vertex of D and adjacent to a vertex of V-D.

RESULTS

Now in the following theorem we established the relationship between our concept with strong split domination and domination number.

Theorem 1: For any connected (p,q) graph G, $\gamma_{rr}[M(G)] + 2 \geq \gamma_{ss}(G) + \gamma(G)$.
Proof : Let A={ $u_1,u_2,u_3, ...u_n$}⊆V(G) such that every vertex of V(G)-A is adjacent to at least one vertex of A and N[A]=V(G). If the induced subgraph <A> is totally disconnected, then A is a γ_{ss} – set of G. Let V_1= {$v_1,v_2,v_3, …..v_n$}⊆V(G) be the set of all nonend vertices in G. Suppose there exists a minimal set of vertices S = {$v_1,v_2,v_3, …..v_k$} ⊆ V_1 such that N[S] = V(G). Then S forms a minimal dominating set of G. Further, let B = { $u_1,u_2,u_3, ...u_n$} ⊆ V[M(G)] be the set of all end vertices of M(G). Now suppose A_1 ⊆ A and every vertex of V[M(G)] – { A_1 U B} is adjacent with at least one vertex of { A_1 U B} and at least one vertex of V[M(G)] – { A_1 U B} such that N[A_1 U B] = V[M(G)], which gives { A_1 U B} is a restrained dominating set of M(G). If the induced graph of < A_1 U B > is regular then {A_1 U B} is a γ_{rr} – set of M(G). It follows that | {A_1 U B} | + 2 ≥ |A| + |S|. Hence $\gamma_{rr}[M(G)] + 2 \geq \gamma_{ss}(G) + \gamma(G)$.

Theorem 2: For any connected (p,q) graph G, $p \geq \gamma_{rr}[M(G)]$.
Proof : Let C= { $u_1,u_2,u_3, ...u_n$} ⊆ V[M(G)] be the set of all end vertices of M(G). Now suppose A_1 ⊂ V[M(G)] such that N[A_1 U C] = V[M(G)]. Also ∀v_i∈V[M(G)] – { A_1 U C} is adjacent to at least one vertex of V[M(G)] – { A_1 U C} and at least one vertex of { A_1 U C}. Then clearly { A_1 U C} is a restrained dominating set of M(G). Suppose the induced subgraph < A_1 U C > is regular. Then { A_1 U C} is a γ_{rr} - set (G). Since |p| = V(G). It follows that |p| ≥ |{ A_1 U C }|. Hence $p \geq \gamma_{rr}[M(G)]$.

Theorem 3: For any connected (p,q) graph G, $q \leq \gamma_{rr}[M(G)] + \beta_1(G) + 2$.
Proof : Let W= { $e_1,e_2,e_3, ...e_n$} = E(G). Suppose W_1 = { $e_1,e_2,e_3, ...e_m$}⊆E(G) be the maximal set of edges with N(e_i) Ⴖ N(e_j) = e and e∈ W-W_1. Clearly, W_1 forms a maximal independent edge set in G.
Further, since V[M(G)] = V(G) U E(G). Let D = { $u_1,u_2,u_3, ...u_n$}⊆V[M(G)] be the γ – set of M(G). Suppose there exists a set A = {$v_1,v_2,v_3, …..v_m$} ⊆ V[M(G)], such that ∀v_i∈A, 1≤ i ≤ m are the vertices with maximum degree. Let D ⊂ V[M(G)] be the set of all end vertices and N[D_1 U D] = V[M(G)]. Clearly { D_1 U D} is a dominating set of M(G). Suppose ∀v_i∈ V(G) – { D_1 U D} is adjacent to at least one vertex of { D_1 U D} and V[M(G)] – { D_1 U D}. If the induced subgraph of <D_1 U D> is regular, then clearly D_1 U D is a γ_{rr} – set of M(G). Since q = E(G). It follows that |q| ≤ | D_1 U D | + |W_1|+ 2, which gives $q \leq \gamma_{rr}[M(G)] + \beta_1(G) + 2$.

Theorem 4: For any connected (p,q) graph G, $\gamma_c(G) + \propto_1(G) \geq \gamma_{rr}[M(G)]$.
Proof : Let B={$v_1,v_2,v_3, …..v_n$}⊆V(G) be the minimal set of vertices which covers all the vertices of G such that N[B]=V(G). Then B is a γ – set of G. Further, if the induced subgraph <B> has exactly one component, then B itself is a connected dominating set of G. Otherwise if B has more than one component, then attach minimum set of vertices {w_i} from V(G) – B which are in u – w path ∀ u, v ∈ V-B gives a single component B_1=BU{w_i}. Clearly B_1 forms a minimal γ_c – set of G.
Suppose A= { $e_1,e_2,e_3, ...e_m$}⊆E(G) be the maximal set of edges with N(e_i)ႶN(e_j)=e, ∀ e_i, e_j ∈ B, 1≤i≤n, 1≤j≤n and e ∈ E(G) – A. Clearly A forms a maximal independent edge set in G. Suppose K= {$v_1,v_2,v_3, …..v_n$} be the set of vertices which are incident with the edges of A and if |K|=p, then K itself is an edge covering number. Otherwise consider the minimum number of edges {e_m}⊆ E(G) – K, such that A_1=KU{e_m} forms a minimal edge covering set of G.
Further, let X={ $u_1,u_2,u_3, ...u_n$} ⊆ V[M(G)] be the set of all nonend vertices of M(G). Now suppose X_1⊆ B and every vertex of V[M(G)]- {X_1UX} is adjacent with at least one vertex of {X_1UX} and at least one vertex of V[M(G)]- {X_1UX} such that N[X_1UX]=V[M(G)], which gives {X_1UX} is a restrained dominating set of M(G). If the induced

subgraph $< X_1 U X >$ is regular, then $\{X_1 U X\}$ is a $\gamma_{rr}[M(G)]$. Hence $|B_1|+|A_1| \geq |\{X_1 U X\}|$ which gives $\gamma_c(G) + \propto_1(G) \geq \gamma_{rr}[M(G)]$.

Theorem 5: For any connected (p,q) graph G, $\gamma_R(G) + \Delta(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let $f : V(G) \rightarrow \{0,1,2\}$ and partition the vertex set of V(G) into $[V_0,V_1,V_2]$ induced by f with $|V_i| = n_i$ for i=0,1,2. Suppose the set V_2 dominates V_0, then $S=V_1 U V_2$ forms a minimal roman dominating set of G.

Further, since V[M(G)] = V(G)UE(G). Suppose there exists $K \subseteq V[M(G)]$ and N[K]=V[M(G)]. Then K is a minimal dominating set of M(G). If for every $v_i \in \{V[M(G)]\text{-}K\}$ is adjacent to at least one vertex of K and at least one vertex of {V[M(G)]-K}, then K is a minimal restrained dominating set of M(G). Assume the induced subgraph < K> is regular. Then K is a regular minimal restrained dominating set of M(G). Since for any graph G, then there exists at least one vertex of maximum degree $v \in V[G]$, such that deg(v) = $\Delta(G)$. Hence $|S| + \Delta(G) \geq |K|$, which gives $\gamma_R(G) + \Delta(G) \geq \gamma_{rr}[M(G)]$.

Theorem 6: For any connected (p,q) graph G, $\gamma_s(G) + \beta_0(G) + \delta(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let $A= \{v_1,v_2,v_3, \ldots..v_n\} \subseteq V(G)$ be the set of all end vertices in G and $A^{'} = V(G) - A$. Suppose there exists a vertex set $D \subset A^{'}$ such that N[D]=V(G). If the induced subgraph <D> has more than one component then D forms a γ_s – set of G.

Let $K= \{ u_1,u_2,u_3, \ldots.u_n\} \subseteq V(G)$ be the minimum set of vertices such that dist(u,v)≥2 and $N(u) \cap N(v)=x$, $\forall u,v \in K$ and $x \in V(G) - K$. Clearly $|K|= \beta_0(G)$.

Since V[M(G)]=V(G)UE(G). Further, let $B= \{v_1,v_2,v_3, \ldots..v_n\} \subseteq V[M(G)]$ be the set of all end vertices in M(G) and $B^{'} = V[M(G)] - B$. Then there exists vertex set $H \subseteq B^{'}$ such that N[HUB]=V[M(G)]. So that {HUB} is a dominating set of V[M(G)]. Since $\forall v_i \in$ [M(G) – {HUB}] is adjacent to at least one vertex of {HUB} and V[M(G)] – {HUB} and the induced subgraph < {HUB} > is regular, then {HUB} is a γ_{rr} - set of M(G). For any graph G, there exists one vertex of minimum degree $v \in V(G)$, such that deg(v)= $\delta(G)$. Since $D \subset V[M(G)]$ and $K \subset V[M(G)]$, then it follows that $|D| + |K| + \delta(G) \geq |\{HUB\}|$. Hence $\gamma_s(G) + \beta_0(G) + \delta(G) \geq \gamma_{rr}[M(G)]$.

Theorem 7: For any connected (p,q) graph G, $\gamma_{cot}(G) + diam(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let $W=\{v_1,v_2,v_3, \ldots..v_n\} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices of G such that N[W]=V(G). Further if the induced subgraph <V(G) - W> has no isolates, then W is a cototal dominating set of G. Otherwise there exists a set H of vertices which are isolates in <V(G) - W> such that {WUH} forms a minimal total dominating set of G. Clearly {WUH} is a minimal cototal dominating set of G.

Let $B \subseteq V(G)$ be the minimal set of vertices. Further, there exists an edge set $J \subseteq J^{'}$, where $J^{'}$ is the set of edges which are incident with the vertices of B constituting the longest path in G such that |J|=diam(G).

Further, let $K= \{v_1,v_2,v_3, \ldots..v_n\} \subset V[M(G)]$ be the set of all end vertices in M(G) and $K_1=V[M(G)]\text{-}K$. Then there exists a vertex set $L \subseteq K_1$ such that $\forall v_i \in$ V[M(G)]-{LUK} is adjacent to at least one vertex of {LUK} and in [V[M(G)] – {LUK}]. Then {LUK} is a γ_{rr} – set of M(G).

It follows that, $|WUH| + |J| \geq |LUK|$. Hence $\gamma_{cot}(G) + diam(G) \geq \gamma_{rr}[M(G)]$.

In [5], given two adjacent vertices u and v we say that u weakly dominates v if deg(u)≤deg(v). A set $D \subseteq V(G)$ is a weak dominating set of G if every vertex in V-D is weakly dominated by atleast one vertex in D. The weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set.

Theorem 8: For any connected (p,q) graph G, $\gamma_w(G) + \propto_0(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let $A= \{v_1,v_2,v_3, \ldots..v_n\} \subseteq V(G)$ be the minimal dominating set of G. If every vertex $u \in V(G)\text{-}A$ is adjacent with $v \in A$ and deg(v)≤deg(u). Then A is a weak dominating set of G.

Suppose $B=\{ u_1,u_2,u_3, \ldots.u_n\} \subseteq V(G)$, $\forall e_i \in E(G)$ is incident to at least one vertex of B. Then $|B|=\propto_0(G)$.

Further, let $K=\{ u_1,u_2,u_3, \ldots.u_n\} \subseteq V[M(G)]$ be the set of all end vertices in M(G) and K'=V[M(G)]-K. Then there exists vertex set $H \subseteq K'$ such that N[HUK]=V[M(G)] so that {HUK} is a dominating set of V[M(G)]. Since $\forall v_i \in$ [M(G)-{HUK}] is adjacent to at least one vertex of {HUK} and V[M(G)]-{HUK}. If the induced subgraph <HUK> is regular, then {HUK} is a γ_{rr} - set M(G). Since $A \subset V[M(G)]$ and $B \subset V[M(G)]$, then it follows that $|A| + |B| \geq |\{HUB\}|$. Which gives $\gamma_w(G) + \propto_0(G) \geq \gamma_{rr}[M(G)]$.

Theorem 9: For any connected (p,q) graph G, $\gamma_{rr}[M(G)] + 1 \geq \gamma_t(G)$.

Proof: Let A= {$v_1, v_2, v_3, \ldots v_n$}⊆ V(G) be the minimal set of vertices which covers all the vertices in G. Clearly A forms a dominating set of G. Suppose the subgraph <A> has no isolates. Then A itself is a γ_t –set of G. Otherwise if deg(v_k)<1 then attach the vertices $w_i \in N(v_k)$ to make deg(v_k)≥1 such that <AU{w_i}> does not contain any isolated vertex. Clearly AU{w_i} forms a total dominating set of G.

Further let B={$v_1, v_2, v_3, \ldots v_k$}⊆ V[M(G)] be the set of all end vertices in M(G) and B_1=V[M(G)]-B. Then there exists vertex set H⊆ B_1 such that ∀ $v_i \in$ V[M(G)]-{HUB} is adjacent to at least one vertex of {HUV} and a vertex of V[M(G)]-{HUB}. Then {HUB} is a γ_r – set of M(G). If the induced subgraph <HUB> is regular, then {HUB} is a γ_{rr} – set of M(G). One can easily see that [AU{w_i}]=V[M(G)]. It follows that |HUB| +1 ≥ |AU{w_i}|. Hence $\gamma_{rr}[M(G)] + 1 \geq \gamma_t(G)$.

A set of edges in a graph G=(V,E) is called an edge dominating set of G if every edge in E-F is adjacent to at least one edge in F. Equivalently, a set F of edges in G is called an edge dominating set of G if for every edge e∈F, there exists an edge e_1∈F such that e and e_1 have a vertex in common. The edge domination number γ'(G) of a graph G is the minimum cardinality of edge dominating set of G[9].

Theorem 10: For any connected (p,q) graph G, $\gamma'(G) + \gamma_{st}(G) \leq \gamma_{rr}[M(G)] + 2$.

Proof: Let A={ $e_1, e_2, e_3, \ldots e_n$}⊆E(G), if for every edge e ∈ E-A then there exists an edge e' ∈ A such that e and e' have a common vertex. Then A is a minimal edge dominating set of G.

Let B={$v_1, v_2, v_3, \ldots v_n$} be the vertex set of G. Suppose B_1⊆B such that N[B_1]=V(G). If deg(u)≥deg(v), ∀ u ∈ B_1 and ∀ v ∈ {B- B_1}, u is adjacent to v. Then B_1 is a strong dominating set of G.

Further, let C= { $u_1, u_2, u_3, \ldots u_n$}⊆V[M(G)] be the set of all end vertices in M(G) and C_1=V[M(G)]-C. Then there exists a vertex set H⊆C_1 such that ∀ $v_i \in$ V[M(G)]-{HUC} is adjacent to at least one vertex of {HUC} and V[M(G)]-{HUC}. If the induced subgraph <HUC> is regular then {HUC} is γ_{rr} – set of M(G). It follows that |A| + |B_1| ≤ |{HUC}| + 2.

Hence $\gamma'(G) + \gamma_{st}(G) \leq \gamma_{rr}[M(G)] + 2$.

Theorem 11: For any connected (p,q) graph G, $\gamma_{rr}[M(G)] + \gamma(G) \geq \gamma_{st}(G) + \propto_1(G)$.

Proof: Suppose C={ $e_1, e_2, e_3, \ldots e_n$}⊆E(G) be the minimal set of edges with N[e_i]∩N[e_j]=e, ∀ e_i, e_j ∈ B, 1≤i≤n, 1≤j≤n and e ∈ E(G)-C. Suppose D={$v_1, v_2, v_3, \ldots v_n$} be the set of vertices which are incident with the edges of C and if |D|=P, then D itself is an edge covering number of G. Since γ(G) ⊆V[M(G)], then γ[M(G)] ⊆ γ_{rr}[M(G)] and then from Theorem 10 the result follows.

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