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PROJECTIVE GEOMETRY

Kartik Rai, Ayush Sharma, Mohit Tanwar
Student, Dronacharya College Of Engineering, Gurgaon

***Abstract*- Computer graphics models are often in 3D. Display devices are 2D, whether LCD screen or printer, so there has to be a projection from 3D to 2D . Projective geometry is the mathematical subject which studies projection. In computer graphics various kinds of projection are possible. In CAD we may want 1-point, 2-point or 3-point projections. In many other applications, we typically want perspective to indicate depth into the scene. In order to understand this properly, we will concentrate first on perspective.**

***Index Terms*- projective lines, projective planes, projective transformation**

I. INTRODUCTION

In mathematics, **projective geometry** is the study of geometric properties that are invariant under projective transformations. This means that, compared to elementary geometry, projective geometry has a different setting, projective space and a selective set of basic geometric concepts. The basic intuitions are that projective space has *more* points than Euclidean space,in a given dimension, and that geometric transformations are permitted that move the extra points to traditional points, and vice versa.

Properties meaningful in projective geometry are respected by this new idea of transformation, which is more radical in its effects than expressible by a transformation matrix and translation. The first issue for geometers is what kind of geometric language is adequate to the novel situation. It is not possible to talk about angles in projective geometry as it is in

Euclidean geometry, because angle is an example of a concept not invariant under projective transformations, as is seen clearly in perspective drawing. One source for projective geometry was indeed the theory of perspective. Another difference from elementary geometry is the way in which parallel lines can be said to meet in a points at infinity, once the concept is translated into projective geometry's terms. Again this notion has an intuitive basis, such as railway tracks meeting at the horizon in a perspective drawing. See projective planes for the basics of projective geometry in two dimension.

II. History

The first geometrical properties of a projective nature were discovered in the third century A.D. by Pappus of Alexandria Filippo Brunelleschi(1404–1472) started investigating the geometry of perspective in 1425 Johannes Kepler (1571–1630) and Gérard Desargues(1591–1661) independently developed the pivotal concept of the "point at infinity".Desargues developed an alternative way of constructing perspective drawings by generalizing the use of vanishing points to include the case when these are infinitely far away. He made Euclidean geometry, where parallel lines are truly parallel, into a special case of an all-encompassing geometric system. Desargues's study on conic sections drew the attention of 16-year old Blaise Pascal and helped him formulate Pascal's theorem. The works of Gaspard Monge at the end of 18th and beginning of 19th century were important for the subsequent development of projective geometry. The work of Desargues was ignored until Michel Chasles chanced upon a handwritten copy in 1845. Meanwhile, Jean-Victor Poncelet had published the foundational treatise on projective geometry in 1822. Poncelet separated the projective properties of objects in individual class and establishing a relationship between metric and projective properties. The non-Euclidean geometries discovered shortly thereafter were eventually demonstrated to have models, such as the Klein model of hyperbolic space, relating to projective geometry.

Paul Dirac studied projective geometry and used it as a basis for developing his concepts of Quantum Mechanics, although his published results were always in algebraic form. See a blog article referring to an article and a book on this subject, also to a talk Dirac gave to a general audience in 1972 in Boston about projective geometry, without specifics as to its application in his physics.

III. PROJECTION LINES

In mathematics, a **projective line** is a one-dimensional projective space. The projective line over a field *K*, denoted $\mathbf{P}^1(K)$, may be defined as the set of one-dimensional subspacesof the two-dimensional vector space K^2 (it does carry other geometric structures)

Homogeneous coordinates:

An arbitrary point in the projective line $\mathbf{P}^1(K)$ may be given in *homogeneous coordinates* by a pair

$$[x_1 : x_2]$$

of points in *K* which are not both zero. Two such pairs are equal if they differ by an overall (nonzero) factor λ:

$$[x_1 : x_2] = [\lambda x_1 : \lambda x_2].$$

Real projective line

The projective line over the real numbers is called the **real projective line**. It may also be thought of as the line *K* together with an idealized *point at infinity* ∞ ; the point connects to both ends of *K* creating a closed loop or topological circle.

An example is obtained by projecting points in $\mathbf{R}^2$ onto the unit circle and then identifying diametrically opposite points. In terms of group theory we can take the quotient by the subgroup {1,−1}.

Compare the extended real number line, which distinguishes ∞ and −∞.

Complex projective line: the Riemann sphere

Adding a point at infinity to the complex plane results in a space that is topologically a sphere. Hence the complex projective line is also known as the **Riemann sphere** (or sometimes the *Gauss sphere*). It is in constant use in complex analysis, algebraic geometry and complex manifold theory, as the simplest example of a compact Riemann surface.

For a finite field

The case of *K* a finite field *F* is also simple to understand. In this case if *F* has *q* elements, the projective line has

$$q + 1$$

elements. We can write all but one of the subspaces as

$$y = ax$$

with *a* in *F*; this leaves out only the case of the line *x* = 0. For a finite field there is a definite loss if the projective line is taken to be this set, rather than an algebraic curve — one should at least see the underlying *infinite* set of points in an algebraic closure as potentially *on* the line.

IV. PROJECTION PLANES

In mathematics, a **projective plane** is a geometric structure that extends the concept of a plane. In the ordinary Euclidean plane, two lines typically intersect in a single point, but there are some pairs of lines (namely, parallel lines) that do not intersect. A projective plane can be thought of as an ordinary plane equipped with additional "points at infinity" where parallel lines intersect. Thus *any* two lines in a projective plane intersect in one and only one point.

renaissance artists, in developing the techniques of drawing in perspective laid the groundwork for this mathematical topic. The archetypical example is the real projective plane , also known as the **extended Euclidean plane**. This example, in slightly different guises, is important in algebraic geometry , topology and projective geometry where it may be denoted variously by PG(2, **R**), $\mathbf{RP}^2$, or $P_2(\mathbf{R})$ among other notations. There are many other projective planes, both infinite, such as the complex projective plane, and finite, such as the fano plane.

A projective plane is a 2-dimensional projective space but not all projective planes can be embedded in 3-dimensional projective spaces. The embedding property is a consequence of a result known as Desargues theorem.

Desargues' theorem and Desarguesian planes

The theorem of Desargues is universally valid in a projective plane if and only if the plane can be constructed from a three-dimensional vector space over a skewfield as above. These planes are called **Desarguesian planes**, named after Gérard Desargues. The real (or complex) projective plane and the projective plane of order 3 given above are examples of Desarguesian projective planes. The projective planes that cannot be constructed in this manner are called non-Desarguesian planes, and the Moulton plane given above is an example of one. The PG(2,*K*) notation is reserved for the Desarguesian planes.

V. PROJECTION TRANSFORMATION

In projective geometry, a **homography** is an isomorphism of projective spaces, induced by an isomorphism of the vector spaces from which they are derived. It is a bijection that maps lines to lines, and thus a collineation. In general, there are collineations which are not homographies, but the fundamental theory of projection geometry asserts that is not so in the case of real projective spaces of dimension at least two. Synonyms include **projectivity**, **projective transformation**, and **projective collineation**.

Historically, homographies (and projective spaces) have been introduced to study perspective and projection in eculidean geometry, and the term "homography", which, etymologically, roughly means "similar drawing" date from this time. At the end of 19th century, formal definitions of projective spaces were introduced, which differed from extending eculidean or affine spaces by adding points at infinity. The term "projective transformation" originated in these abstract constructions. These constructions divide into two classes that have been shown to be equivalent. A projective space may be constructed as the set of the lines of a vector space over a given field (the above definition is based on this version); this construction facilitates the definition of projective coordinates and allows using the tools of linear algebra for the study of homographies. The alternative approach consists in defining the projective space through a set of axioms, which do not involve explicitly any field in this context, collineations are easier to define than homographies, and homographies are defined as specific collineations, thus called "projective collineations".

For sake of simplicity, unless otherwise stated, the projective spaces considered in this article are supposed to be defined over a (commutative) field. Equivalently pappus's hexagon theorem and Desargues' theorem are supposed to be true. A large part of the results remain true, or may be generalized to projective geometries for which these theorems do not hold.

Definition and expression in homogeneous coordinates

A projective space *P*(*V*) of dimension *n* over a field *K* may be defined as the set of the lines in a *K*-vector space of dimension *n*+1. If a basis of *V* has been fixed, a point of *V* may be represented by a point $(x_0, \ldots, x_n)$ of K^{n+1}. A point of *P*(*V*), being a line in *V*, may thus be represented by the coordinates of any nonzero point of this line, which are thus called homogeneous coordinates of the projective point.

Given two projective spaces *P*(*V*) and *P*(*W*) of the same dimension, an **homography** is a mapping from *P*(*V*) to *P*(*W*), which is induced by an isomorphism of vector spaces $f : V \to W$. Such an isomorphism induces a bijection from *P*(*V*) to *P*(*W*), because of the linearity of *f*. Two such isomorphisms, *f* and *g*, define the same homography if and only if there is a nonzero element *a* of *K* such that *g* = *af*.

This may be written in terms of homogeneous coordinates in the following way: A homography φ may be defined by a nonsingular *n*+1 × *n*+1 matrix $[a_{i,j}]$, called the *matrix of the homography*. This matrix is defined up to the multiplication by a nonzero element of *K*. The homogeneous coordinates $[x_0 : \cdots : x_n]$ of a point and the coordinates $[y_0 : \cdots : y_n]$ of its image by φ are related by

$$y_0 = a_{0,0}x_0 + \cdots + a_{0,n}x_n$$
$$\vdots$$
$$y_n = a_{n,0}x_0 + \cdots + a_{n,n}x_n.$$

When the projective spaces are defined by adding points at infinity to affine spaces (projective completion) the preceding formulas become, in affine coordinates,

$$y_1 = \frac{a_{1,0} + a_{1,1}x_1 + \cdots + a_{1,n}x_n}{a_{0,0} + a_{0,1}x_1 + \cdots + a_{0,n}x_n}$$

$$\vdots$$

$$y_n = \frac{a_{n,0} + a_{n,1}x_1 + \cdots + a_{n,n}x_n}{a_{0,0} + a_{0,1}x_1 + \cdots + a_{0,n}x_n}$$

which generalizes the expression of the homographic function of the next section. This defines only a partial function between affine spaces, which is defined only outside the hyperplane where the denominator is zero.

VI. CONCLUSIONS

When possible, use domain and task knowledge to choose model:

• What type of information is needed

• What aspects of the imaging conditions are known or controlled

• What types of uncertainty can be modeled and compensated for

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