PROPOSITIONAL CALCULUS

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ABSTRACT- **The propositional calculus (PC) is a formal language that adequately represents the set of valid (truth preserving) inferences which depend on coordinate expressions such as *and, or, not, if…then…, if and only if*. From the optic of PC, we are only interested in those inferences whose validity depends on the role of these expressions.**

$$F \Rightarrow (G \Rightarrow F) \quad (1)$$

$$(F \Rightarrow G) \Rightarrow ((F \Rightarrow (G \Rightarrow H)) \Rightarrow (F \Rightarrow H)) \quad (2)$$

$$F \Rightarrow (G \Rightarrow F \wedge G) \quad (3)$$

$$F \Rightarrow F \vee G \quad (4)$$

$$F \Rightarrow G \vee F \quad (5)$$

$$F \wedge G \Rightarrow F \quad (6)$$

$$F \wedge G \Rightarrow G \quad (7)$$

$$(F \Rightarrow G) \Rightarrow ((H \Rightarrow G) \Rightarrow (F \vee H \Rightarrow G)) \quad (8)$$

$$(F \Rightarrow G) \Rightarrow ((F \Rightarrow \neg G) \Rightarrow \neg F) \quad (9)$$

Although it is possible to construct an abstract formal calculus that has no immediate practical use and next to nothing in the way of obvious applications, the very name *calculus* indicates that this species of formal system owes its origin to the utility of its prototypical members in practical calculation. Generally speaking, any mathematical calculus is designed with the intention of representing a given domain of formal objects, and typically with the aim of facilitating the computations and inferences that need to be carried out in this representation. Thus some idea of the intended denotation, the formal objects that the formulas of the calculus are intended to denote, is given in advance of developing the calculus itself.

INDEX TERMS : Calculus , prototypical

I. INTRODUCTION

In mathematical logic, a **propositional calculus** or **logic** (also called **sentential calculus** or **sentential logic**) is a formal system in which formulas of a formal language may be interpreted to represent propositions. A system of inference rules and axioms allows certain formulas to be derived. These derived formulas are called theorems and may be interpreted to be true propositions. A constructed sequence of such formulas is known as a *derivation* or *proof* and the last formula of the sequence is the theorem. The derivation may be interpreted as proof of the proposition represented by the theorem. Usually in **Truth-functional propositional logic**, formulas are interpreted as having either a truth value of *true* or a truth value of *false* Truth-functional propositional logic and systems isomorphic to it, are considered to be **zeroth-order logic**.

Our propositional calculus has ten inference rules. These rules allow us to derive other true formulas given a set of formulas that are assumed to be true. The first eight simply state that we can infer certain wffs from other wffs. The last two rules however use hypothetical reasoning in the sense that in the premise of the rule we temporarily assume an (unproven) hypothesis to be part of the set of inferred formulas to see if we can infer a certain other formula. Since the first eight rules don't do this they are usually described as *non-hypothetical* rules, and the last two as *hypothetical* rules.

Double negative elimination

From the *wff* $\neg\neg\varphi$, we may infer φ

Conjunction introduction

From any *wff* φ and any *wff* ψ, we may infer $(\varphi \wedge \psi)$.

Conjunction elimination

From any *wff* $(\varphi \wedge \psi)$, we may infer φ and ψ

Disjunction introduction

From any *wff* φ, we may infer $(\varphi \vee \psi)$ and $(\psi \vee \varphi)$, where ψ is any *wff*.

Disjunction elimination

From the *wff*s of the form $(\varphi \vee \psi)$, $(\varphi \rightarrow \chi)$, and $(\psi \rightarrow \chi)$, we may infer χ.

Biconditional introduction

From the *wff*s of the form $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$, we may infer $(\varphi \leftrightarrow \psi)$.

Biconditional elimination

From the *wff* $(\varphi \leftrightarrow \psi)$, we may infer $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$.

Modus ponens

From the *wff*s of the form φ and $(\varphi \rightarrow \psi)$, we may infer ψ.

Conditional proof

If ψ can be derived while assuming the hypothesis φ, we may infer $(\varphi \rightarrow \psi)$.

Reductio ad absurdum

If we can derive both ψ and $\neg\psi$ while assuming the hypothesis φ, we may infer $\neg\varphi$.

[1] Example of a proof

The following is an example of a (syntactical) demonstration:

Prove: $A \rightarrow A$

Proof:

Number	wff	Justification
1	A	p
2	$A \vee A$	From (1) by disjunction introduction
3	$(A \vee A) \wedge A$	From (1) and (2) by conjunction introduction
4	A	From (3) by conjunction elimination
5	$A \vdash A$	Summary of (1) through (4)
6	$\vdash A \rightarrow A$	From (5) by conditional proof

Interpret $A \vdash A$ as "Assuming A, infer A". Read $\vdash A \rightarrow A$ as "Assuming nothing, infer that A implies A," or "It is a tautology that A implies A," or "It is always true that A implies A."

II. BASIC CALCULUS

The following outlines a standard propositional calculus. Many different formulations exist which are all more or less equivalent but differ in the details of:

1. their language, that is, the particular collection of primitive symbols and operator symbols,
2. the set of axioms, or distinguished formulas, and
3. the set of inference rules.

Any given proposition may be represented with a letter called a 'propositional constant', analogous to representing a number by a letter in mathematics, for instance, $a = 5$. All propositions require exactly one of two truth-values: true or false. For example, let P be the proposition that it is raining outside. This will be true (P) if it is raining outside and false otherwise ($\neg P$).

- We then define truth-functional operators, beginning with negation. ($\neg P$ represents the negation of P, which can be thought of as the denial of P. In the example above, ($\neg P$ expresses that it is not raining outside, or by a more standard reading: "It is not the case that it is raining outside." When P is true, ($\neg P$ is false; and when P is false, ($\neg P$ is true. {($\neg\neg P$ always has the same truth-value as P.
- Conjunction is a truth-functional connective which forms a proposition out of two simpler propositions, for example, P and Q. The conjunction of P and Q is written $P \wedge Q$, and expresses that each are true. We read $P \wedge Q$ for "P and Q". For any two propositions, there are four possible assignments of truth values:

1. P is true and Q is true
2. P is true and Q is false
3. P is false and Q is true
4. P is false and Q is false

The conjunction of P and Q is true in case 1 and is false otherwise. Where P is the proposition that it is raining outside and Q is the proposition that a cold-front is over Kansas, $P \wedge Q$ is true when it is raining outside and there is a cold-front over Kansas. If it is not raining outside, then $P \wedge Q$ is false; and if there is no cold-front over Kansas, then $P \wedge Q$ is false.

- Disjunction resembles conjunction in that it forms a proposition out of two simpler propositions. We write it $P \vee Q$, and it is read "P or Q". It expresses that either P or Q is true. Thus, in the cases listed above, the disjunction of P and Q is true in all cases except 4. Using the example above, the disjunction expresses that it is either raining outside or there is a cold front over Kansas. (Note, this use of disjunction is supposed to resemble the use of the English word "or". However, it is most like the English inclusive "or", which can be used to express the truth of at least one of two propositions. It is not like the English exclusive"or", which expresses the truth of exactly one of two propositions. That is to say, the exclusive "or" is false when both P and Q are true (case 1). An example of the exclusive or is: You may have a bagel or a pastry, but not both. Often in natural language, given the appropriate context, the addendum "but not both" is omitted but implied. In mathematics, however, "or" is always inclusive or; if exclusive or is meant it will be specified, possibly by "xor".)
- Material conditional also joins two simpler propositions, and we write $P \rightarrow Q$, which is

read "if P then Q". The proposition to the left of the arrow is called the antecedent and the proposition to the right is called the consequent. (There is no such designation for conjunction or disjunction, since they are commutative operations.) It expresses that Q is true whenever P is true. Thus it is true in every case above except case 2, because this is the only case when P is true but Q is not. Using the example, if P then Q expresses that if it is raining outside then there is a cold-front over Kansas. The material conditional is often confused with physical causation. The material conditional, however, only relates two propositions by their truth-values—which is not the relation of cause and effect. It is contentious in the literature whether the material implication represents logical causation.

- Biconditional joins two simpler propositions, and we write $P \leftrightarrow Q$, which is read "P if and only if Q". It expresses that P and Q have the same truth-value, thus P if and only if Q is true in cases 1 and 4, and false otherwise.

It is extremely helpful to look at the truth tables for these different operators, as well as the method of analytic tableaux.

Closure under operations

Propositional logic is closed under truth-functional connectives. That is to say, for any proposition φ, $\neg\varphi$ is also a proposition. Likewise, for any propositions φ and ψ, $\varphi \wedge \psi$ is a proposition, and similarly for disjunction, conditional, and biconditional. This implies that, for instance, $\varphi \wedge \psi$ is a proposition, and so it can be conjoined with another proposition. In order to represent this, we need to use parentheses to indicate which proposition is conjoined with which. For instance, $P \wedge Q \wedge R$ is not a well-formed formula, because we do not know if we are conjoining $P \wedge Q$ with R or if we are conjoining P with $Q \wedge R$. Thus we must write either $(P \wedge Q) \wedge R$ to represent the former, or $P \wedge (Q \wedge R)$ to represent the latter. By evaluating the truth conditions, we see that both expressions have the same truth conditions (will be true in the same cases), and moreover that any proposition formed by arbitrary conjunctions will have the same truth conditions, regardless of the location of the parentheses. This means that conjunction is associative, however, one should not assume that parentheses never serve a purpose. For instance, the sentence $P \wedge (Q \vee R)$ does not have the same truth conditions of $(P \wedge Q) \vee R$, so they are different sentences distinguished only by the parentheses. One can verify this by the truth-table method referenced above.

Note: For any arbitrary number of propositional constants, we can form a finite number of cases which list their possible truth-values. A simple way to generate this is by truth-tables, in which one writes P, Q, ..., Z, for any list of k propositional constants—that is to say, any list of propositional constants with k entries. Below this list, one writes 2^k rows, and below P one fills in the first half of the rows with true (or T) and the second half with false (or F). Below Q one fills in one-quarter of the rows with T, then one-quarter with F, then one-quarter with T and the last quarter with F. The next column alternates between true and false for each eighth of the rows, then sixteenths, and so on, until the last propositional constant varies between T and F for each row. This will give a complete listing of cases or truth-value assignments possible for those propositional constants.

Argument

The propositional calculus then defines an *argument* to be a set of propositions. A valid argument is a set of propositions, the last of which follows from—or is implied by—the rest. All other arguments are invalid. The simplest valid argument is modus ponens, one instance of which is the following set of propositions:

$$\begin{array}{ll} 1. & P \rightarrow Q \\ 2. & P \\ \hline \therefore & Q \end{array}$$

This is a set of three propositions, each line is a proposition, and the last follows from the rest. The first two lines are called premises, and the last line the conclusion. We say that any proposition C follows from any set of propositions $(P_1, ..., P_n)$, if C must be true whenever every member of the set $(P_1, ..., P_n)$ is true. In the argument above, for any P and Q, whenever $P \rightarrow Q$ and P are true, necessarily Q is true. Notice that, when P is true, we cannot consider cases 3 and 4 (from the truth table). When $P \rightarrow Q$ is true, we cannot consider case 2. This leaves only case 1, in which Q is also true. Thus Q is implied by the premises.

This generalizes schematically. Thus, where φ and ψ may be any propositions at all,

$$\begin{array}{ll} 1. & \varphi \to \psi \\ 2. & \varphi \\ \hline \therefore & \psi \end{array}$$

Other argument forms are convenient, but not necessary. Given a complete set of axioms (see below for one such set), modus ponens is sufficient to prove all other argument forms in propositional logic, thus they may be considered to be a derivative. Note, this is not true of the extension of propositional logic to other logics like first-order logic. First-order logic requires at least one additional rule of inference in order to obtain completeness. The significance of argument in formal logic is that one may obtain new truths from established truths. In the first example above, given the two premises, the truth of *Q* is not yet known or stated. After the argument is made, *Q* is deduced. In this way, we define a deduction system to be a set of all propositions that may be deduced from another set of propositions. For instance, given the set of propositions $A = \{P \vee Q, \neg Q \wedge R, (P \vee Q) \to R$, we can define a deduction system, Γ, which is the set of all propositions which follow from *A*. Reiteration is always assumed, so $P \vee Q, \neg Q \wedge R, (P \vee Q) \to R \in \Gamma$. Also, from the first element of *A*, last element, as well as modus ponens, *R* is a consequence, and so $R \in \Gamma$. Because we have not included sufficiently complete axioms, though, nothing else may be deduced. Thus, even though most deduction systems studied in propositional logic are able to deduce $(P \vee Q) \leftrightarrow (\neg P \to Q)$, this one is too weak to prove such a proposition.

Propositional calculus is about the simplest kind of logical calculus in any current use. (Aristotelian "syllogistic" calculus, which is largely supplanted in modern logic, is in *some* ways simpler--but in other ways more complex--than propositional calculus.) It can be extended in several ways.

The most immediate way to develop a more complex logical calculus is to introduce rules that are sensitive to more fine-grained details of the sentences being used. When the "atomic sentences" of propositional logic are broken up into terms, variables, predicates, and quantifiers, they yield first-order logic, or first-order predicate logic, which keeps all the rules of propositional logic and adds some new ones. (For example, from "All dogs are mammals" we may infer "If Rover is a dog then Rover is a mammal.) With the tools of first-order logic it is possible to formulate a number of theories, either with explicit axioms or by rules of inference, that can themselves be treated as logical calculi. Arithmetic is the best known of these; others include set theory and mereology. Modal logic also offers a variety of inferences that cannot be captured in propositional calculus. For example, from "Necessarily *p*" we may infer that *p*. From *p* we may infer "It is possible that *p*". Many-valued logics are those allowing sentences to have values other than *true* and *false*. (For example, *neither* and *both* are standard "extra values"; "continuum logic" allows each sentence to have any of an infinite number of "degrees of truth" between *true* and *false*.) These logics often require calculational devices quite distinct from propositional calculus.

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