© 2014 IJIRT | Volume 1 Issue 6 | ISSN : 2349-6002

CONVOLUTION AND APPLICATIONS OF CONVOLUTION

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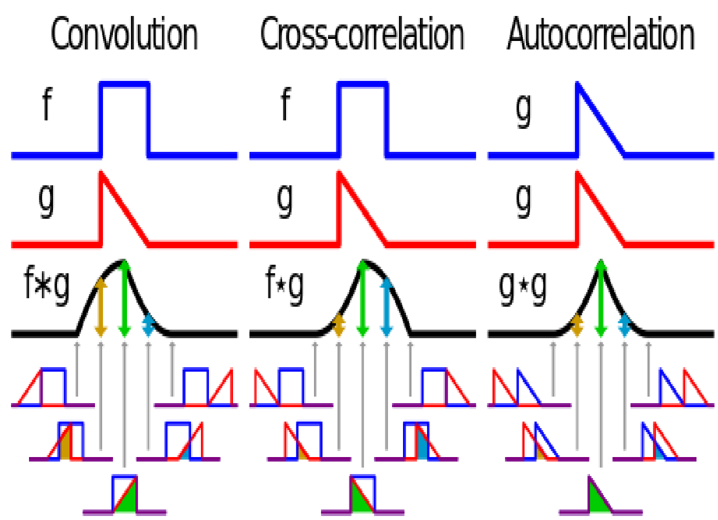
Abstract-We introduce an integral transform related to a Fourier sine-Fourier - Fourier cosine generalized convolution and prove a Watson type theorem for the transform. As applications we obtain solutions of some integral equations in closed form.. Convolution describes the output (in terms of the input) of an important class of operations known as *linear time-invariant* (LTI). See LTI system theory for a derivation of convolution as the result of LTI constraints. In terms of the Fourier transforms of the input and output of an LTI operation, no new frequency components are created. The existing ones are only modified (amplitude and/or phase). In other words, the output transform is the pointwise product of the input transform with a third transform (known as a transfer function). See Convolution theorem for a derivation of that property of convolution. Conversely, convolution can be derived as the inverse Fourier transform of the pointwise product of two Fourier transforms.

***Index Terms*-Convolution, Watson theorem, Fourier sine transform, fourier cosine transform, Integral equation, Hölder inequality**

I. INTRODUCTION

In mathematics and, in particular, functional analysis, **convolution** is a mathematical operation on two functions *f* and *g*, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. Convolution is similar to cross-correlation. It has applications that include probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations.



The convolution of *f* and *g* is written *f*∗*g*, using an asterisk or star. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$(f*g)(t) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$

(commutativity)

While the symbol *t* is used above, it need not represent the time domain. But in that context, the convolution formula can be described as a weighted average of the function *f*(*τ*) at the moment *t* where the weighting is given by *g*(−*τ*) simply shifted by amount *t*. As *t* changes, the weighting function emphasizes different parts of the input function.

For functions *f*, *g* supported on only $[0, \infty)$(i.e., zero for negative arguments), the integration limits can be truncated, resulting in

$$(f*g)(t) = \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

II. HISTORICAL DEVELOPMENTS

One of the earliest uses of the convolution integral appeared in D'Alembert's derivation of Taylor's theorem in *Recherches sur différents points importants du système du monde*, published in 1754.[1]

Also, an expression of the type:

$$\int f(u)\cdot g(x-u)\, du$$

is used by Sylvestre François Lacroix on page 505 of his book entitled *Treatise on differences and series*, which is the last of 3 volumes of the encyclopedic series: *Traité du calcul différentiel et du calcul intégral*, Chez Courcier, Paris, 1797-1800.[2] Soon thereafter, convolution operations appear in the works of Pierre Simon Laplace, Jean Baptiste Joseph Fourier, Siméon Denis Poisson, and others. The term itself did not come into wide use until the 1950s or 60s. Prior to that it was sometimes known as *faltung* (which means *folding* in German), *composition product*, *superposition integral*, and *Carson's integral*.[3] Yet it appears as early as 1903, though the definition is rather unfamiliar in older uses.[4][5]

The operation:

$$\int_0^t \varphi(s)\psi(t-s)\,ds, \qquad 0 \le t < \infty$$

is a particular case of composition products considered by the Italian mathematician Vito Volterra in 1913.[6]

III. CIRCULAR CONVOLUTION

When a function g_T is periodic, with period T, then for functions, f, such that $f*g_T$ exists, the convolution is also periodic and identical to:

$$(f * g_T)(t) \equiv \int_{t_0}^{t_0+T} \left[\sum_{k=-\infty}^{\infty} f(\tau + k$$

where t_o is an arbitrary choice. The summation is called a periodic summation of the function *f*.

When g_T is a periodic summation of another function, *g*, then $f*g_T$ is known as a *circular* or *cyclic* convolution of *f* and *g*.

And if the periodic summation above is replaced by f_T, the operation is called a *periodic* convolution of f_T ~~and g_T.~~

IV. DISCRETE CONVOLUTION

For complex-valued functions *f*, *g* defined on the set **Z** of integers, the **discrete convolution** of *f* and *g* is given by:[7]

$$(f * g)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m]\, g[n-m]$$

$$= \sum_{m=-\infty}^{\infty} f[n-m]\, g[m].$$

(commutativity)

The convolution of two finite sequences is defined by extending the sequences to finitely supported functions on the set of integers. When the sequences are the coefficients of two polynomials, then the coefficients of the ordinary product of the two polynomials are the convolution of the original two sequences. This is known as the Cauchy product of the coefficients of the sequences.

Thus when *g* has finite support in the set $\{-M, -M+1, \dots, M-1, M\}$ (representing, for instance, a finite impulse response), a finite summation may be used:[8]

$$(f * g)[n] = \sum_{m=-M}^{M} f[n-m]g[m].$$

V. CIRCULAR DISCRETE CONVOLUTION

When a function g_N is periodic, with period *N*, then for functions, *f*, such that $f*g_N$ exists, the convolution is also periodic and identical to:

$$(f * g_N)[n] \equiv \sum_{m=0}^{N-1} \left(\sum_{k=-\infty}^{\infty} f[m + kN]\right) g$$

The summation on *k* is called a periodic summation of the function *f*.

If g_N is a periodic summation of another function, *g*, then $f*g_N$ is known as a circular convolution of *f* and *g*.

When the non-zero durations of both *f* and *g* are limited to the interval $[0, N-1]$, $f*g_N$ reduces to these common forms:

$$(f * g_N)[n] = \sum_{m=0}^{N-1} f[m]\ g_N[n$$

$$= \sum_{m=0}^{n} f[m]\ g[n -$$

(*Eq. 1*)

$$= \sum_{m=0}^{N-1} f[m]\ g[(n$$

The notation ($f *_N g$) for *cyclic convolution* denotes convolution over the cyclic group of integers modulo *N*.

Circular convolution arises most often in the context of fast convolution with an FFT algorithm.

VI. INTEGRABLE FUNCTIONS

The convolution of *f* and *g* exists if *f* and *g* are both Lebesgue integrable functions in $L^1(\mathbf{R}^d)$, and in this case $f*g$ is also integrable (Stein & Weiss 1971, Theorem 1.3). This is a consequence of Tonelli's theorem. This is also true for functions in ℓ^1, under the discrete convolution, or more generally for the convolution on any group.

Likewise, if $f \in L^1(\mathbf{R}^d)$ and $g \in L^p(\mathbf{R}^d)$ where $1 \le p \le \infty$, then $f*g \in L^p(\mathbf{R}^d)$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

In the particular case $p = 1$, this shows that L^1 is a Banach algebra under the convolution (and equality of the two sides holds if f and g are non-negative almost everywhere).

More generally, Young's inequality implies that the convolution is a continuous bilinear map between suitable L^p spaces. Specifically, if $1 \le p,q,r \le \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

then

$$\|f * g\|_r \le \|f\|_p \|g\|_q, \quad f \in \mathcal{L}^p,\ g$$

so that the convolution is a continuous bilinear mapping from $L^p \times L^q$ to L^r. The Young inequality for convolution is also true in other contexts (circle group, convolution on **Z**).

The preceding inequality is not sharp on the real line: when $1 < p, q, r < \infty$, there exists a constant $B_{p,q} < 1$ such that:

$$\|f * g\|_r \le B_{p,q} \|f\|_p \|g\|_q, \quad f \in \mathcal{L}$$

The optimal value of $B_{p,q}$ was discovered in 1975.[13]

A stronger estimate is true provided $1 < p, q, r < \infty$:

$$\|f * g\|_r \le C_{p,q} \|f\|_p \|g\|_{q,w}$$

where $\|g\|_{q,w}$ is the weak L^q norm. Convolution also defines a bilinear continuous map $L^{p,w} \times L^{q.w} \to L^{r,w}$ for $1 < p, q, r < \infty$, owing to the weak Young inequality:[14]

$$\|f * g\|_{r,w} \le C_{p,q} \|f\|_{p,w} \|g\|_{r,w}.$$

VII. PROPERTIES

7.1 Algebraic properties

The convolution defines a product on the linear space of integrable functions. This product satisfies the following algebraic properties, which formally mean that the space of integrable functions with the product given by convolution is a commutative algebra without identity (Strichartz 1994, §3.3). Other linear spaces of functions, such as the space of continuous functions of compact support, are closed under the convolution, and so also form commutative algebras.

Commutativity

$$f * g = g * f$$

Associativity

$$f * (g * h) = (f * g) * h$$

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Associativity with scalar multiplication

$$a(f * g) = (af) * g$$

for any real (or complex) number a.

Multiplicative identity

No algebra of functions possesses an identity for the convolution. The lack of identity is typically not a major inconvenience, since most collections of functions on which the convolution is performed can be convolved with a delta distribution or, at the very least (as is the case of L^1) admit approximations to the identity. The linear space of compactly supported distributions does, however, admit an identity under the convolution. Specifically,

$$f * \delta = f$$

where δ is the delta distribution.

Inverse element

Some distributions have an inverse element for the convolution, $S^{(-1)}$, which is defined by

$$S^{(-1)} * S = \delta.$$

The set of invertible distributions forms an abelian group under the convolution.

Complex conjugation

$$\overline{f * g} = \overline{f} * \overline{g}$$

7.3 Integration

If f and g are integrable functions, then the integral of their convolution on the whole space is simply obtained as the product of their integrals:

$$\int_{\mathbf{R}^d} (f * g)(x)\,dx = \left(\int_{\mathbf{R}^d} f(x)\,dx\right)\left(\int_{\mathbf{R}^d}\right.$$

This follows from Fubini's theorem. The same result holds if f and g are only assumed to be nonnegative measurable functions, by Tonelli's theorem.

7.2 Differentiation

In the one-variable case,

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

where d/dx is the derivative. More generally, in the case of functions of several variables, an analogous formula holds with the partial derivative:

$$\frac{\partial}{\partial x_i}(f * g) = \frac{\partial f}{\partial x_i} * g = f * \frac{\partial g}{\partial x_i}.$$

A particular consequence of this is that the convolution can be viewed as a "smoothing" operation: the convolution of f and g is differentiable as many times as f and g are in total.

These identities hold under the precise condition that f and g are absolutely integrable and at least one of them has an absolutely integrable (L^1) weak derivative, as a consequence of Young's inequality. For instance, when f is continuously differentiable with compact support, and g is an arbitrary locally integrable function,

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g.$$

These identities also hold much more broadly in the sense of tempered distributions if one of *f* or *g* is a compactly supported distribution or a Schwartz function and the other is a tempered distribution. On the other hand, two positive integrable and infinitely differentiable functions may have a nowhere continuous convolution.

In the discrete case, the difference operator $D f(n) = f(n + 1) - f(n)$ satisfies an analogous relationship:

$$D(f * g) = (Df) * g = f * (Dg).$$

VIII. APPLICATIONS

Convolution and related operations are found in many applications in science, engineering and mathematics.

- In image processing

See also: digital signal processing

In digital image processing convolutional filtering plays an important role in many important algorithms in edge detection and related processes.

In optics, an out-of-focus photograph is a convolution of the sharp image with a lens function. The photographic term for this is bokeh.

In image processing applications such as adding blurring.

- In digital data processing

In analytical chemistry, Savitzky–Golay smoothing filters are used for the analysis of spectroscopic data. They can improve signal-to-noise ratio with minimal distortion of the spectra.

In statistics, a weighted moving average is a convolution.

- In acoustics, reverberation is the convolution of the original sound with echos from objects surrounding the sound source.

In digital signal processing, convolution is used to map the impulse response of a real room on a digital audio signal.

In electronic music convolution is the imposition of a spectral or rhythmic structure on a sound. Often this envelope or structure is taken from another sound. The convolution of two signals is the filtering of one through the other.

- In electrical engineering, the convolution of one function (the input signal) with a second function (the impulse response) gives the output of a linear time-invariant system (LTI). At any given moment, the output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence (expressed as a multiplicative factor). The impulse response function provides that factor as a function of the elapsed time since each input value occurred.
- In physics, wherever there is a linear system with a "superposition principle", a convolution operation makes an appearance. For instance, in spectroscopy line broadening due to the Doppler effect on its own gives a Gaussian spectral line shape and collision broadening alone gives a Lorentzian line shape. When both effects are operative, the line shape is a convolution of Gaussian and Lorentzian, a Voigt function.

In Time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses, and the measured fluorescence is a sum of exponential decays from each delta pulse.

In computational fluid dynamics, the large eddy simulation (LES) turbulence model uses the convolution operation to lower the range of length scales necessary in computation thereby reducing computational cost.

- In probability theory, the probability distribution of the sum of two independent random variables is the convolution of their individual distributions.

In kernel density estimation, a distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian. (Diggle 1995).

- In radiotherapy treatment planning systems, most part of all modern codes of calculation applies a convolution-superposition algorithm.

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