LINEAR PROGRAMMING

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***Abstract*-This paper considers the basic of linear programming. It will give us brief idea about linear programming. We describe the types of problems Linear Programming can**

handle and show how we can solve them using the simplex method. We discuss generalizations to Binary Integer Linear Programming

I. INTRODUCTION

Linear programming is the process of taking various linear inequalities relating to some situation, and to find the "best" value obtainable under these conditions. A typical example would be taking the limitations of materials and labor, and then determining the "best" production levels for maximal profits under those conditions.

In "real life", linear programming is part of a very important area of mathematics called "optimization techniques". This field of study are used every day in the organization and allocation of resources. These "real life" systems can have dozens or hundreds of variables, or more. In algebra, though, you'll only work with the simple (and graphable) two-variable linear case.

The general process for solving linear-programming exercises is to graph the inequalities (called the "constraints") to form a walled-off area on the *x*,*y*-plane (called the "feasibility region"). Then we figure out the coordinates of the corners of the feasibility region , and test these corner points in the formula (called the "optimization equation") for which we are trying to find the highest or lowest value.

II. STANDARD FORM

Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following three parts:

A linear function to be maximized

e.g. $f(x_1, x_2) = c_1 x_1 + c_2 x_2$

Problem constraints of the following form

e.g.

$$a_{11}x_1 + a_{12}x_2 \le b_1$$
$$a_{21}x_1 + a_{22}x_2 \le b_2$$
$$a_{31}x_1 + a_{32}x_2 \le b_3$$

Non-negative variables

e.g.

$$x_1 \ge 0$$
$$x_2 \ge 0$$

The problem is usually expressed in *matrix form*, and then becomes:

$$\max\{c^{\mathrm{T}}x \mid Ax \le b \wedge x \ge 0\}$$

Other forms, such as minimization problems, problems with constraints on alternative forms, as well as problems involving negative variables can always be rewritten into an equivalent problem in standard form.

Example

Suppose that a farmer has a piece of farm land, say *L* km^2, to be planted with either wheat or barley or some combination of the two. The farmer has a limited amount of fertilizer, *F* kilograms, and insecticide, *P* kilograms. Every square kilometer of wheat requires F_1 kilograms of fertilizer, and P_1 kilograms of insecticide, while every square kilometer of barley requires F_2 kilograms of fertilizer, and P_2 kilograms of insecticide. Let S_1 be the selling price of wheat per square kilometer, and S_2 be the selling price of barley. If we denote the area of land planted with wheat and barley by x_1 and x_2 respectively, then profit can be maximized by choosing optimal values for x_1 and x_2. This problem can be expressed with the following linear programming problem in the standard form:

Maximize: $S_1 \cdot x_1 + S_2 \cdot x_2$

Subject to: $x_1 + x_2 \le L$

$$F_1 \cdot x_1 + F_2 \cdot x_2 \le F$$

$$P_1 \cdot x_1 + P_2 \cdot x_2 \le P$$
$$x_1 \ge 0, x_2 \ge 0$$

Which in matrix form becomes:

maximize $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

subject to

$$\begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} L \\ F \\ P \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

III. DUALITY

Every linear programming problem, referred to as a *primal* problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. In matrix form, we can express the *primal* problem as:

Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$;

with the corresponding **symmetric** dual problem,

Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \ge \mathbf{c}$, $\mathbf{y} \ge 0$.

An alternative primal formulation is:

Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$;

with the corresponding **asymmetric** dual problem,

Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} = \mathbf{c}$, $\mathbf{y} \ge 0$.

There are two ideas fundamental to duality theory. One is the fact that (for the symmetric dual) the dual of a dual linear program is the original primal linear program. Additionally, every feasible solution for a linear program gives a bound on the optimal value of the objective function of its dual. The weak duality theorem states that the objective function value of the dual at any feasible solution is always greater than or equal to the objective function value of the primal at any feasible solution. Thestrong duality theorem states that if the primal has an optimal solution, $\mathbf{x}^*$, then the dual also has an optimal solution, $\mathbf{y}^*$, and $\mathbf{c}^T\mathbf{x}^*=\mathbf{b}^T\mathbf{y}^*$.

A linear program can also be unbounded or infeasible. Duality theory tells us that if the primal is unbounded then the dual is infeasible by the weak duality theorem. Likewise, if the dual is unbounded, then the primal must be infeasible. However, it **is possible** for both the dual and the primal to be infeasible.

Example

Sometimes, one may find it more intuitive to obtain the dual program without looking at the program matrix. Consider the linear programing:

minimize $\sum_{i=1}^{m} c_i x_i + \sum_{j=1}^{n}$

subject to $\sum_{i=1}^{m} a_{ij} x_i + e_j \;,\; 1 \le j \le n$

$$f_i x_i + \sum_{j=1}^{n} b_{ij} \;,\; 1 \le i \le m$$

$$x_i \ge 0,\; t_j \ge \;,\; 1 \le i \le m, 1$$

We have $m + n$ conditions and all variables are non-negative. We shall define $m + n$ dual variables: $\mathbf{y}_j$ and $\mathbf{s}_i$. We get:

minimize $\sum_{i=1}^{m} c_i x_i + \sum_{j=1}^{n} d_j t_j$

subject to $\sum_{i=1}^{m} a_{ij} x_i \cdot y_j + e_j \;,\; 1 \le j \le n$

$$f_i x_i \cdot s_i + \sum_{j=1}^{n} b_{ij} \;,\; 1 \le i \le m$$

$$x_i \ge 0,\; t_j \ge 0 \quad ,\; 1 \le i \le m,$$

$$y_j \ge 0,\; s_i \ge 0 \quad ,\; 1 \le j \le n,$$

Since this is a minimization problem, we would like to obtain a dual program that is a lower bound of the primal. In other words, we would like the sum of all right hand side of the constraints to be the maximal under the condition that for each primal variable the sum of its coefficients do not exceed its coefficient in the linear function. For example, $\mathbf{x}_1$ appears in $n + 1$ constraints. If we sum its constraints' coefficients we get $a_{1,1}\mathbf{y}_1 + a_{1,2}\mathbf{y}_2 + ... + a_{1,n}\mathbf{y}_n + f_1\mathbf{s}_1$. This sum must be at most $\mathbf{c}_1$. As a result we get:

maximize $\sum_{j=1}^{n} g_j y_j + \sum_{i=1}^{m}$

subject to $\sum_{j=1}^{n} a_{ij} y_j + f_i \ , \ 1 \le i \le m$

$$e_j y_j + \sum_{i=1}^{m} b_{ij} \ , \ 1 \le j \le n$$

$$y_j \ge 0, \ s_i \ge \ , \ 1 \le j \le n, 1 \le$$

Note that we assume in our calculations steps that the program is in standard form. However, any linear program may be transformed to standard form and it is therefore not a limiting factor.

IV. INTEGRAL LINEAR PROGRAMS

A linear program in real variables is said to be integral if it has at least one optimal solution which is integral. Likewise, a polyhedron $P = \{x \mid Ax \ge 0\}$ is said to be integral if for all bounded feasible objective functions c, the linear program $\{\max cx \mid x \in P\}$ has an optimum x^* with integer coordinates. As observed by Edmonds and Giles in 1977, one can equivalently say that the polyhedron P is integral if for every bounded feasible integral objective function c, the optimal value of the linear program $\{\max cx \mid x \in P\}$ is an integer.

Integral linear programs are of central importance in the polyhedral aspect of combinatorial optimization since they provide an alternate characterization of a problem. Specifically, for any problem, the convex hull of the solutions is an integral polyhedron; if this polyhedron has a nice/compact description, then we can efficiently find the optimal feasible solution under any linear objective. Conversely, if we can prove that a linear programming relaxation is integral, then it is the desired description of the convex hull of feasible (integral) solutions.

One common way of proving that a polyhedron is integral is to show that it is totally unimodular. There are other general methods including the integer decomposition property and total dual integrality. Other specific well-known integral LPs include the matching polytope, lattice polyhedra, submodular flow polyhedra, and the intersection of 2 generalized polymatroids/g-polymatroids --- e.g. see Schrijver 2003.

A bounded integral polyhedron is sometimes called a convex lattice polytope, particularly in two dimensions.

V. USES

Linear programming is a considerable field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Certain special cases of linear programming, such as *network flow* problems and *multicommodity flow* problems are considered important enough to have generated much research on specialized algorithms for their solution. A number of algorithms for other types of optimization problems work by solving LP problems as sub-problems. Historically, ideas from linear programming have inspired many of the central concepts of optimization theory, such as *duality,decomposition,* and the importance of *convexity* and its generalizations. Likewise, linear programming is heavily used in microeconomics and company management, such as planning, production, transportation, technology and other issues. Although the modern management issues are ever-changing, most companies would like to maximize profits or minimize costs with limited resources. Therefore, many issues can be characterized as linear programming problems.

VI. CONCLUSION

we have only touched the beginning of a very important generalization
of Linear Programming. Once we know how to do and
solve Linear Programming problems, it is tempting to convert other problems to Linear
Programming problems.

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