An Introduction to Linear Programming

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Abstract- **This paper will address you about the many practical problems in operations can be expressed as linear programming problems. Certain special cases of linear programming, such as network flow problems and multi commodity flow problems are considered important enough to have generated much research on specialized algorithms for their solution. A number of algorithms for other types of optimization problems work by solving LP problems as sub-problems. Historically, ideas from linear programming have inspired many of the central concepts of optimization theory, such as duality, decomposition, and the importance of convexity and its generalizations. Likewise, linear programming is heavily used in microeconomics and company management, such as planning, production, transportation, technology and other issues. Although the modern management issues are ever-changing, most companies would like to maximize profits or minimize costs with limited resources. Therefore, many issues can be characterized as linear programming problems.**

I. INTRODUCTION

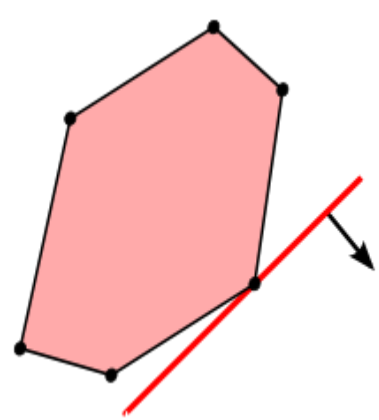
Linear programming (**LP**; also called **linear optimization**) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (mathematical optimization).

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. It's feasible region is a convex polytope, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine function defined on this polyhedron. A linear programming algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such a point exists.

Linear programs are problems that can be expressed in canonical form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ \text{and} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Where x represents the vector of variables (to be determined), **c** and **b** are vectors of (known) coefficients, *A* is a (known) matrix of coefficients, and $(\cdot)^{\mathrm{T}}$ is the matrix transpose. The expression to be maximized or minimized is called the *objective function* ($\mathbf{c}^{\mathrm{T}}\mathbf{x}$ in this case). The inequalities $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ are the constraints which specify a convex polypore over which the objective function is to be optimized. In this context, two vectors are comparable when they have the same dimensions. If every entry in the first is less-than or equal-to the corresponding entry in the second then we can say the first vector is less-than or equal-to the second vector.



A pictorial representation of a simple linear program with two variables and six in equalities. The set of feasible solutions is depicted in light red and forms a polygon, a 2-dimensional polytope. The linear cost function is represented by the red line and the arrow: The red line is a level set of the cost function, and the arrow indicates the direction in which we are optimizing.

II. STANDARD FORM

Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following three parts:

- A **linear function to be maximized**

e.g. $f(x_1, x_2) = c_1 x_1 + c_2 x_2$

- **Problem constraints** of the following form

e.g.

$a_{11}x_1 + a_{12}x_2 \leq b_1$
$a_{21}x_1 + a_{22}x_2 \leq b_2$
$a_{31}x_1 + a_{32}x_2 \leq b_3$

- **Non-negative variables**

e.g.

$x_1 \geq 0$
$x_2 \geq 0$

The problem is usually expressed in matrix form, and then becomes:

$\max\{c^{\mathbf{T}}x \mid Ax \leq b \wedge x \geq 0\}$ Other forms, such as minimization problems, problems with constraints on alternative forms, as well as problems involving negative variables can always be rewritten into an equivalent problem in standard form.

Example:

Suppose that a farmer has a piece of farm land, say L km^2, to be planted with either wheat or barley or some combination of the two. The farmer has a limited amount of fertilizer, F kilograms, and insecticide kilograms. Every square kilometer of wheat requires F_1 kilograms of fertilizer and P_1 kilograms of insecticide, while every square kilometer of barley requires F_2 kilograms of fertilizer and P_2 kilograms of insecticide. Let S_1 be the selling price of wheat per square kilometer, and S_2 be the selling price of barley. If we denote the area of land planted with wheat and barley by x_1 and x_2 respectively, then profit can be maximized by choosing optimal values for x_1 and x_2. This problem can be expressed with the following linear programming problem in the standard form:

imize: $S_1 \cdot x_1 + S_2 \cdot x_2$ ximize the revenue—revenue is the "objective function")

ject to: $+ x_2 \leq L$ it on total area)

$\cdot x_1 + F_2 \cdot x_2 \leq F$ it on fertilizer)

$\cdot x_1 + P_2 \cdot x_2 \leq P$ it on insecticide)

$\geq 0, x_2 \geq 0$ not plant a negative area).

Which in matrix form becomes?

Maximize $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Subject to

$$\begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} L \\ F \\ P \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- **Augmented form** (slack form)

Linear programming problems can be converted into an *augmented form* in order to apply the common form of the simplex algorithm. This form introduces non-negative *slack variables* to replace inequalities with equalities in the constraints. The problems can then be written in the following block matrix form:

Maximize Z:

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$\mathbf{X}, \mathbf{x}_s \geq 0$

Where $\mathbf{x}_s$ are the newly introduced slack variables, and Z is the variable to be maximized.

Example:

The example above is converted into the following augmented form:

imize: $S_1 \cdot x_1 + S_2 \cdot x_2$ ective function)

ject to: $+ x_2 + x_3 = L$ mented constraint)

$\cdot x_1 + F_2 \cdot x_2 + x_4 = F$ mented constraint)

$\cdot x_1 + P_2 \cdot x_2 + x_5 = P$ mented constraint)

$, x_2, x_3, x_4, x_5 \geq 0$.

Where x_3, x_4, x_5 are (non-negative) slack variables, representing in this example the unused area, the amount of unused fertilizer, and the amount of unused insecticide.

In matrix form this becomes:

Maximize *Z*:

$$\begin{bmatrix} 1 & -S_1 & -S_2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & F_1 & F_2 & 0 & 1 & 0 \\ 0 & P_1 & P_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ L \\ F \\ P \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq 0.$$

III. CONCLUSION

In Linear Programming we maximize or minimize a linear function subject to linear constraints. It is a problem solving technique that has been developed to help managers make decisions. Applications of linear programming include the following:

A manufacturing company wants to develop a production schedule that will maximize profits. This objective can also be presented as a production schedule that will minimize cost. A Marketing analyst wants to allocate her fixed advertising budget among alternative media (radio, TV, newspapers) in a manner which will maximize sales of her company's product.

REFERENCE

- ✓ Books:
  - ➢ Mathematical Engineering
  - ➢ Linear Programming in Math
- ✓ Sites:
  - ➢ www.google.com
  - ➢ www.wiki.in