REVIEW ON MAXWELL'S EQUATIONS AND PLANE EM WAVES

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***Abstract*- Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and mathematician James Clerk Maxwell, who published an early form of those equations between 1861 and 1862.**

I. INTRODUCTION

The equations introduce the electric field E, a vector field, and the magnetic field B, a pseudovector field, where each generally have time-dependence. The sources of these fields are electric charges and electric currents, which can be expressed

as local densities namely charge density ρ and current density J. A separate law of nature, the Lorentz force law, describes how the electric and magnetic field act on charged particles and currents.

II. DIELECTRIC AND CONDUCTOR

Displacement vector:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

Polarization vector:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} P_k}{\Delta v}$$

$$V = \frac{1}{4\pi\varepsilon_0} \iiint_{v'} \frac{\vec{P} \cdot \hat{a}_R}{R^2} dv' \quad ,$$

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$\nabla' = \hat{x} \frac{\partial}{\partial x'} + \hat{y} \frac{\partial}{\partial y'} + \hat{z} \frac{\partial}{\partial z'}$$

$$\Rightarrow \nabla' \left(\frac{1}{R} \right) = \frac{\hat{a}_R}{R^2}$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0} \iiint_{v'} \vec{P} \cdot \nabla' \left(\frac{1}{R} \right) dv'$$

$$= \frac{1}{4\pi\varepsilon_0} \left[\iiint_{v'} \nabla' \cdot \left(\frac{\vec{P}}{R} \right) dv' - \iiint_{v'} \frac{\nabla' \cdot \vec{P}}{R} dv' \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \left[\oint\!\!\oint_{s'} \frac{\vec{P} \cdot \hat{a}_n'}{R} d\vec{S}' + \oint\!\!\oint\!\!\oint_{v'} \frac{-\nabla' \cdot \vec{P}}{R} dv' \right]$$

Surface charge density: $\rho_{ps} = \vec{P} \cdot \hat{a}_n$.

Volume charge density: $\rho_p = -\nabla \cdot \vec{P}$

Total charge:

$$Q = \oiint_{s'} \vec{P} \cdot \hat{a}_n \, d\vec{S}' + \iiint_{v'} \nabla \cdot \vec{P} \, dv' = 0$$

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\left(\rho + \rho_p\right) \quad \Rightarrow \nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho$$

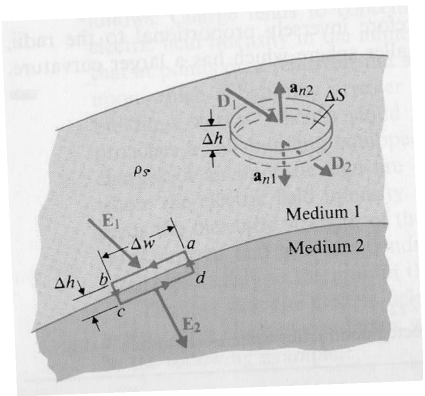
Define $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$$\Rightarrow \nabla \cdot \vec{D} = \rho \Leftrightarrow \oiint_s \vec{D} \cdot d\vec{S} = Q$$

Hall voltage:

Note: Generally, $\vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E}$ or

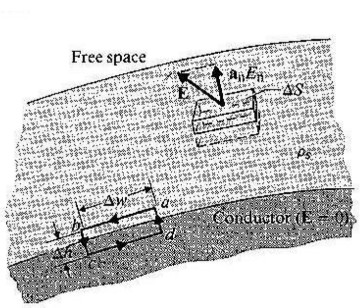
$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$



For biaxial dielectric, $\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$

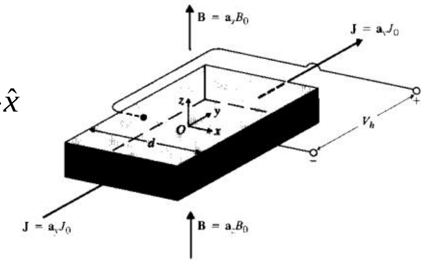
Hall Effect:

Current density: $\vec{J} = \hat{y} J_0 = Nq\vec{v}$



If the material is a conductor or an *n*-type semiconductor the charge carrier are electrons: $q < 0$

Hall field:



$$\vec{E}_h = -\vec{v} \times \vec{B} = -(\hat{y}v_0) \times (\hat{z}B_0) = -\hat{x}$$

Hall coefficient: $C_h = \frac{E_x}{J_y B_z} = \frac{1}{Nq} < 0$

If the material is a *p*-type semiconductor, the charge carries are holes: $q > 0$

Hall field: $\vec{E}_h = \hat{x} v_0 B_0$

Hall voltage: $V_h = -v_0 B_0 d$

Hall coefficient: $C_h > 0$

2-2 Boundary Conditions of Electromagnetic Fields

Boundary conditions for electric fields:

Eg. Show that E_t=0 on the conductor plane.

(Proof) ∵ The E-field inside a conductor is zero,

$$\therefore \oint \vec{E} \cdot d\vec{l} = E_t \Delta W = 0 \Rightarrow E_t = 0$$

$$\oiint \vec{E} \cdot d\vec{S} = E_n \Delta S = \frac{\rho_s \Delta S}{\varepsilon_0} \quad \Rightarrow E_n = \frac{\rho_s}{\varepsilon_0}$$

Eg. Show that $E_{1t} = E_{2t}$ and $\hat{a}_{n2} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s$ on the interface between two dielectric.

(Proof) $\oint_{abcda} \vec{E} \cdot d\vec{l} = E_{1t} \Delta W - E_{2t} \Delta W = 0$,

$E_{1t} = E_{2t}$

$$\oiint_s \vec{D}\cdot d\vec{S} = \left(\vec{D}_1\cdot\hat{a}_{n2} + \vec{D}_2\cdot\hat{a}_{n1}\right)\Delta S = \hat{a}_{n2}\cdot\left(\vec{D}_1 - \vec{D}_2\right)\Delta S = \rho_s \Delta S$$

$$\hat{a}_{n2}\cdot\left(\vec{D}_1 - \vec{D}_2\right) = \rho_s$$ or D_{1n}-D_{2n}=ρ_s

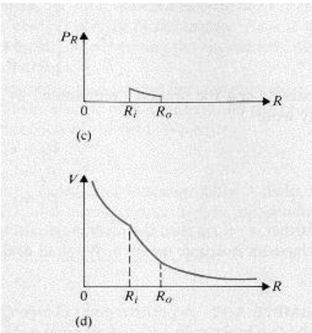
If ρ_s=0, then D_{1n}=D_{2n} or $\varepsilon_1 E_{1n}$=$\varepsilon_2 E_{2n}$

Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

Boundary Conditions between Two Lossless Media

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$
$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$
$$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$
$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

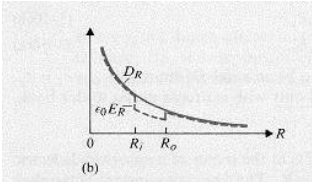


Eg. A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ε_r. Determine $\vec{E}$, V, $\vec{D}$, and $\vec{P}$ as functions of the radial distance R. [高考]

(Sol.) $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$

R>R_o:

$$\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0 R^2}$$, $$V = \frac{Q}{4\pi\varepsilon_0 R}$$



$$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2} \text{ and } \vec{P} = 0$$

R_i<R<R_o:

$$\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2} = \hat{a}_R \frac{Q}{4\pi\varepsilon R^2},$$

$$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}, \quad \vec{P} = \hat{a}_R\left(1 - \frac{1}{\varepsilon_r}\right)\frac{Q}{4\pi R^2}$$

$$V = -\int_\infty^{R_o} \frac{Q}{4\pi\varepsilon_0 R^2} dR - \int_{R_o}^{R} \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2} dR$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r R}\right]$$

R<R_i: $$\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0 R^2}, \quad \vec{D} = \hat{a}_R \frac{Q}{4\pi R^2},$$

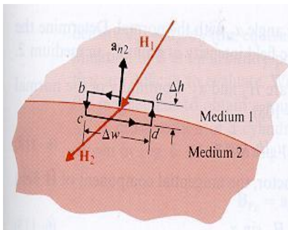
$$\vec{P} = 0,$$

$$V = V\Big|_{R = R_i} - \int_{R_i}^{R} \frac{Q}{4\pi\varepsilon_0 R^2} dR$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$

Boundary conditions for magnetic fields:

Eg. Show that $\mu_1 H_{1n}$=$\mu_2 H_{2n}$ and $\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}$.



(Proof)

$$\oiint \vec{B}\cdot d\vec{S} = 0 \Rightarrow$$

$B_{1n}\Delta S - B_{2n}\Delta S = 0$, B_{1n}=B_{2n}

$\Rightarrow \mu_1 H_{1n}$=$\mu_2 H_{2n}$

$$V_h = -\int_0^d E_h dx = v_0 B_0 d$$

$$\oint \vec{H}\cdot \mathrm{d}\vec{l} = I$$

$$\Rightarrow \oint_{abcda} \vec{H}\cdot \mathrm{d}\vec{l} = H_1 \cdot \Delta w + H_2 \cdot (-\Delta w) = J_{sw}\Delta w$$

$$\Rightarrow H_{1t} - H_{2t} = J_{sw}$$

$$\Rightarrow \hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}$$

If J=0, then H_{1t}=H_{2t}

2-4 Maxwell's Equations and Plane EM Waves

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E}\cdot d\ell = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H}\cdot d\ell = I + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot ds$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D}\cdot ds = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B}\cdot ds = 0$	No isolated magnetic charge

Note: $\frac{\partial \vec{D}}{\partial t}$ is equivalent to a current density, called the displacement current density.

Lorentz condition: $\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t}$ =0

$$\nabla \times \vec{B} = \mu\vec{J} + \mu\frac{\partial \vec{D}}{\partial t} = \nabla\times\nabla\times\vec{A} = \mu\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}(-\nabla V - \frac{\partial \vec{A}}{\partial t}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \nabla(\mu\varepsilon\frac{\partial V}{\partial t}) - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

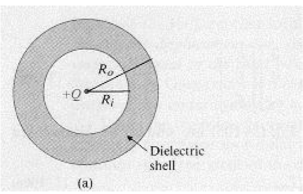
$$\Rightarrow \nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \nabla(\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t})$$

If Lorentz Condition holds, we have

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

∵

$$\nabla\cdot\vec{D} = \rho = \nabla\cdot\varepsilon\vec{E} = \rho = \nabla\cdot\varepsilon(-\nabla V - \frac{\partial\vec{A}}{\partial t}) \Rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \nabla^2 V + \frac{\partial}{\partial t}(-\mu\varepsilon\frac{\partial V}{\partial t}) = -\frac{\rho}{\varepsilon}$$



∴

$$\nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Effective permittivity:

$$\nabla\times\vec{H} = \vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega(\varepsilon + \frac{\sigma}{j\omega})\vec{E} = j\omega\varepsilon_C\vec{E}$$

$$\Rightarrow \varepsilon_C = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon'' \Rightarrow \sigma = \omega\varepsilon''$$.

Similarly, $\mu = \mu' - j\mu''$

Loss tangent: $\tan\delta_C = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$

2-5 Conclusion:

1. Maxwell's Equations (Point Form):

$\nabla \times H = J + \partial D/\partial t$ $\nabla \cdot B = 0$

$\nabla \times E = -\partial B/\partial t$ $\nabla \cdot D = \rho V$

2. Maxwell's Equations (Word Form):

- Magnetic fields circulate around currents and changing electric flux densities.
- There are no magnetic charges (monopoles) in the universe.
- A change in magnetic flux excites a voltage around the flux perimeter.

3. Quantities and Units

Variable	Units	Technical Name
E	Volts/m	Electric Field
H	Amps/m	Magnetic Field
D	Coulombs/m2	Electric Flux Density

B	Webers/m2
Magnetic Flux Density	
J	Amps/m2
Current Density	
ρV	Coulombs/m3
Charge Density (Volume)	

Note:

Also be able to recognize the various forms of Ampere's Law, Faraday's Law,Coulomb's Law,Gauss's Law, the Biot-Savart Law, the Vector Wave Equation, and the Scalar Wave Equation.

REFERENCES:

1. Taflove A & Hagness S C (2005). *Computational Electrodynamics: The Finite-difference Time-domain Method*. Boston MA: Artech House. Chapters 6 & 7.
2. H Freistühler & G Warnecke (2001Hyperbolic problems, numerics (1996).
3. "The Origin of Spurious Solutions in Computational Electromagnetics". *Journal of Computational Physics* 125 (1): 104.
4. J.D. Jackson. "6.11". *Classical Electrodynamics* (3rd ed.).
5. Maxwell Equations. Ieeeghn.org. Retrieved 2008-10-19.
6. Thomas B. A. Senior & John Leonidas Volakis (1995-03-01).Approximate boundary conditions in electromagnetic. . London UK: Institution of Electrical Engineers. p. 261 ff.
7. T Hagstrom (Björn Engquist & Gregory A. Kriegsmann, Eds.) (1997) Computational wave propagation. Berlin: Springer. p. 1 ff.