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FOURIER SERIES

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Abstract- **Fourier seriesof a function f in a system of functions {ϕn} which are orthonormal on an interval (a,b).2010 Mathematics Subject Classification: Primary: 42A Secondary: 42B [MSN][ZBL]**

***Index Terms* - Euler coefficient, Half range series, Even and odd function**

I. INTRODUCTION

The series
$\sum_{k=0}^{\infty} c_k \phi_k$
whose coefficients are determined by
$c_k=\int_a^b f(x)\phi_k(x)dx, k=1,2,\dots(1)$
These coefficients are called the Fourier coefficients of f. In general it is assumed that f is square integrable on (a,b). For many systems $\{\phi_k\}$ this requirement can be relaxed by replacing it by another which ensures the existence of all the integrals in (1). The Fourier series in the trigonometric system is defined for every function f that is integrable on $(0,2\pi)$. It is the series
$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos kx+b_k \sin kx)(2)$
with coefficients
$a_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, b_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx.(3)$

II. PROCEDURE AND VAST DESCRIPTION OF FOURIER SERIES

Fourier series for functions in several variables are constructed analogously. A further generalization leads to Fourier coefficients and Fourier series for elements of a Hilbert space.The theory of Fourier series in the trigonometric system has been most thoroughly developed, and these were the first examples of Fourier series. If one has in mind Fourier series in the trigonometric system, it is usual to talk simply of Fourier series, without indicating the system by which they are constructed.Fourier series form a considerable part of the theory of trigonometric series. Fourier series first appeared in the papers of J. Fourier (1807) devoted to an investigation of the problems of heat conduction. He suggested representing a function f given on $(0,2\pi)$ by the trigonometric series (2) with coefficients determined by (3). Such a choice of coefficients is natural from many points of view. For example, if the series (2) converges uniformly to f, then term-by-term integration leads to the expressions for the coefficients a_k and b_k given in (3). These formulas had been obtained already by L. Euler (1777) by term-by-term integration.Using (3) the Fourier series (2) can be constructed for every function that is integrable over $[0,2\pi]$. Integrability of the function can be understood in various senses, for example integrability according to Riemann or Lebesgue. Depending on this, one speaks of Fourier–Riemann series, Fourier–Lebesgue series, etc. The concepts of the Riemann and the Lebesgue integral themselves arose to a considerable extent in connection with research on Fourier series. The modern presentation of the theory of Fourier series was developed after the construction of the Lebesgue integral, and since then it has developed mainly as the theory of Fourier–Lebesgue series. Below it is assumed that the function f has period 2π and is Lebesgueintegrable over the period.In the theory of Fourier series one studies the relation between the properties of functions and the properties of their Fourier series; in particular, one investigates questions on the representation of functions by Fourier series.The proof of a minimum property of the partial sums of Fourier series goes back to the work of F. Bessel (1828): Given an $f\in L_2$, then among all the trigonometric polynomials of order n,
$t_n(x)=A_0+\sum_{k=0}^{n}(A_k \cos kx+B_k \sin kx)$,
the smallest value of the integral
$\frac{1}{\pi}\int_0^{2\pi}(f(x)-t_n(x))^2dx$
is attained for the partial sum of the Fourier series (2) off:
$s_n(f,x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k \cos kx+b_k \sin kx)$.
This smallest value is equal to
$\frac{1}{\pi}\int_0^{2\pi} f^2(x)dx-\left(\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\right)$.
This implies the Bessel inequality
$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\le\frac{1}{\pi}\int_0^{2\pi} f^2(x)dx$,

which is satisfied for every function f in L2.

The system of trigonometric functions is a closed system (cf. Closed system of elements (functions)), that is, if $f \in L_2$, then the Parseval equality

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)=\frac{1}{\pi}\int_0^{2\pi} f^2(x)dx$$

is valid, where a_k, b_k are the Fourier coefficients of f. In particular, for functions f in L2 the series

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2) \quad (4)$$

is convergent. The converse assertion also holds: If for a system of numbers a_k, b_k the series (4) converges, then these numbers are the Fourier coefficients of a certain function $f \in L_2$ (F. Riesz and E. Fischer, 1907).

The Fourier coefficients of any integrable function tend to zero. This statement is called the Riemann–Lebesgue theorem. B. Riemann proved it for Fourier–Riemann series and H. Lebesgue for Fourier–Lebesgue series.

If the function f is absolutely continuous, then the Fourier series for the derivative f′ can be obtained by term-by-term differentation of the Fourier series for f. This implies that if the derivative of order $r \geq 0$ of a function f is absolutely continuous, then the estimates

$$a_k, b_k = o(k^{-(r+1)}), \quad k \to \infty,$$

are valid for the Fourier coefficients off.

The first convergence criterion for Fourier series was obtained by P.G.L. Dirichlet in 1829. His result (the Dirichlet theorem) can be formulated as follows: If a function f has a finite number of maxima and minima over the period and is everywhere continuous, except at a finite number of points where it may have discontinuities of the first kind, then the Fourier series of fconverges for all x, and, moreover, at points of continuity it converges to f(x) and at points of discontinuity it converges to$(f(x+0)+f(x-0))/2$. Subsequently, this assertion was extended to arbitrary functions of bounded variation (C. Jordan, 1881).According to the localization principle proved by Riemann (1853), the convergence or divergence of the Fourier series of a function f at a point x, and the value of the sum when it converges, depends only on the behaviour of f in an arbitrarily small neighbourhood of x.Many different convergence criteria for Fourier series at a point are known. R. Lipschitz (1864) established that the Fourier series of a function f converges at a point x if $|f(x+h)-f(x)| \leq M|h|^\alpha$ is satisfied for all sufficiently small h, where M and α are certain positive constants (the Lipschitz criterion). The Dini criterion is more general: The Fourier series of a function f converges to S at a point x if the integral

$$\int_0^{\pi} \frac{|\phi_x(t)|}{t} dt$$

converges, where $\phi_x(t)=f(x+t)+f(x-t)-2S$. The value f(x) is usually taken for S. For example, if the Fourier series of f converges at a point x where this function is continuous, then the sum of the series is necessarily equal to f(x).

Lebesgue (1905) proved that if

$$\int_0^h |\phi_x(t)|dt=o(h), \quad \int_h^{\pi} \frac{|\phi_x(t+h)-\phi_x(t)|}{t} dt=o(1),$$

as $h \to 0$, then the Fourier series of f converges to S at x. This Lebesgue criterion is stronger than all those given above and stronger than the De la Vallée-Poussin criterion and the Young criterion. But verifying it is usually difficult.A convergence criterion of another type is given by the Hardy–Littlewood theorem (1932): The Fourier series of a function f converges at a point x if the following conditions are satisfied:

1)

$$f(x+h)-f(x)=o(\ln^{-1}\frac{1}{|h|}),$$

as $h \to 0$; and

2) the estimates

$$a_k, b_k = O(k^{-\delta}), \quad \delta>0,$$

are valid for the Fourier coefficients off.

Besides convergence criteria for Fourier series at a point, criteria for uniform convergence have been studied also. Let a function f have period 2π and be continuous. Then its Fourier series converges uniformly to it on the whole real line if the modulus of continuity $\omega(f,\delta)$ off satisfies the condition

$$\omega(f,\delta)\ln\delta \to 0 \text{ as } \delta \to 0$$

(the Dini–Lipschitz criterion) or if f has bounded variation (the Jordan criterion).

From this one can obtain criteria for uniform convergence of Fourier series on a certain interval if the localization principle for uniform convergence is used. The latter is formulated as follows. If two functions are equal on an interval [a,b], then on each strictly interior interval $[a+\epsilon, b-\epsilon]$, $\epsilon>0$, either the Fourier series of these functions are both uniformly convergent or neither is uniformly convergent. In other words, the uniform convergence of the Fourier series of a function f on an interval depends only on the behaviour of f in an arbitrarily small extension of this interval. Bois-Reymond (1876) established that the continuity of a function at a certain point does not guarantee that its Fourier series converges at this point. Later it was proved that the Fourier series of a continuous function may diverge on an everywhere-dense set of measure zero that is of the second category.If nothing is assumed about the function except that it is integrable, then its Fourier series may turn out to be divergent almost-everywhere, or even everywhere. The first examples of such functions were constructed by A.N. Kolmogorov (1923, 1926). Later it was shown that this may be true both for the Fourier series of the function itself and for the function conjugate to it.

As early as 1915, N.N. Luzin made the conjecture that the Fourier series of every L_2-function converges almost-everywhere. For a long time only partial results were obtained in this direction. The general form of the problem turned out to be very difficult and it was only in 1966 that L. Carleson proved the validity of this conjecture (see Carleson theorem). The Fourier series of L_p-functions when $p>1$ also converge almost-everywhere. Kolmogorov's example shows that it is impossible to strengthen this result any further in terms of the spaces L_p.

Since the partial sums of a Fourier series do not always converge, one also considers the summation of Fourier series by some average of the partial sums and uses this to represent the function. One of the simplest examples are the Fejér sums, which are the arithmetical means of the partial sums $s_k(f,x)$ of the Fourier series:

$$\sigma_n(f,x)=\frac{1}{n+1}\sum_{k=0}^{n}s_k(f,x).$$

For every integrable function f the sums $\sigma_n(f,x)$ converge to $f(x)$ almost-everywhere and, moreover, converge at every point where f is continuous; if f is continuous everywhere, then they converge uniformly.

According to the Denjoy–Luzin theorem, if the trigonometric series (2) at every x converges absolutely on a set of positive measure, then the series

$$\sum_k(|a_k|+|b_k|) \quad (5)$$

converges, and hence the series (2) converges absolutely for all x. Thus, the absolute convergence of (2) is equivalent to convergence of (5).

S.N. Bernstein [S.N. Bernshtein] (1934) proved that if the modulus of continuity $\omega(f,\delta)$ of a function f satisfies

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\omega(f,1/n)<\infty,$$

then the Fourier series of f converges absolutely. It is impossible to weaken this condition: If $\omega(\delta)$ is a modulus of continuity of function type such that the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\omega(1/n)$$

diverges, then a function f can be found with modulus of continuity satisfying $\omega(f,\delta)\leq\omega(\delta)$ and whose Fourier series does not converge absolutely.

In particular, the Fourier series of functions satisfying a Lipschitz condition of order $\alpha>1/2$ converge absolutely. When $\alpha=1/2$, absolute convergence need not hold (Bernshtein, 1914).

If f is a function of bounded variation and if its modulus of continuity satisfies

$$\sum_{n=1}^{\infty}\frac{1}{n}\sqrt{\omega(f,1/n)}<\infty, \quad (6)$$

then the Fourier series of f converges absolutely (see [Sa]). Condition (6) cannot be weakened (see [Bo]).

In contrast to the above, the following theorem gives a criterion for the absolute convergence for an individual function. A necessary and sufficient condition for the absolute convergence of the Fourier series of a function f is that the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}e_n(f)$$

converges, where $e_n(f)$ is the best approximation to f in the metric of L_2 by trigonometric polynomials containing n harmonics (see [St]).

The series (2) can be considered as the real part of the power series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k-ib_k)e^{ikx}.$$

The imaginary part

$$\sum_{k=1}^{\infty}(-b_k\cos kx+a_k\sin kx) \quad (7)$$

is called the series conjugate to the series (2).

Let $f\in L_2$ and let (2) be its Fourier series. Then for almost-all x the function

$$\tilde{f}(x)=\lim_{\epsilon\to+0}\frac{1}{\pi}\int_{\epsilon}^{\pi}\frac{f(x-t)-f(x+t)}{2\tan t/2}dt$$

exists (I.I. Privalov, 1919). The function $\tilde{f}$ is called the conjugate function to f; it need not be integrable. However, if $\tilde{f}\in L_1$, then the Fourier series of $\tilde{f}$ is the series (7) (V.I. Smirnov, 1928).

In many cases one can deduce some property or other of the conjugate series (7) from the properties of the function f or its Fourier series (2), for example, convergence in the metric of L_p, convergence or summability at a point, or almost-everywhere, etc.

Properties of Fourier series under special assumptions on their coefficients have also been studied. For example, lacunary trigonometric series, when the only non-zero coefficients are those indexed by numbers n_m forming a lacunary sequence, that is, $n_{m+1}/n_m \geq \lambda > 1$. Another example of special series are series with monotone coefficients.

All that has been said above concerns Fourier series of the form (2). For Fourier series in a rearranged trigonometric system certain properties of the Fourier series in the trigonometric system, taken in the usual order, do not hold. For example, there is a continuous function such that its Fourier series after a certain rearrangement diverges almost-everywhere (see [KoMe], [Za], [Ul2], [Ol]);

The theory of Fourier series for functions in several variables (multiple Fourier series) has been developed to a lesser extent. Some of the multi-dimensional results are analogous to the one-dimensional results. But there are crucial differences.

Let $x=(x_1,\dots,x_N)$ be a point of the N-dimensional space, let $k=(k_1,\dots,k_N)$ be an N-dimensional vector with integer coordinates and let $(k,x)=k_1x_1+\dots+k_Nx_N$. For a function $f(x)$ with period 2π in each variable and Lebesgueintegrable over the N-dimensional cube $[0,2\pi]^N$, the Fourier series in the trigonometric system is

$$\sum_k c_k e^{i(k,x)}, \quad (8)$$

where the summation is over all k and

$$c_k=\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} f(x)e^{-i(k,x)}dx$$

are the Fourier coefficients of f. The Fourier series (8) is written in complex form. Writing it in trigonometric form as a series in the products of multiple cosines and sines is rather more clumsy.

Various definitions of the partial sums of the series (8) are possible; for example, partial sums over rectangles

$$\sum_{|k_1|\leq n_1}\cdots\sum_{|k_N|\leq n_N} c_k e^{i(k,x)},$$

and over circles

$$\sum_{|k|\leq n} c_k e^{i(k,x)}, \quad (9)$$

where n is the radius and $|k|=\sqrt{k_1^2+\dots+k_N^2}$.

The circular partial sums (9) are not so suitable for representing functions as are their Riesz means

$$\sum_{|k|\leq n}(1-|k|/n)^\alpha c_k e^{i(k,x)}.$$

For Riesz means of order $\alpha\geq(N-1)/2$ of Fourier series of L_2-functions the localization principle is valid; this is not so for smaller α (S. Bochner, 1936). The Riesz means of circular partial sums of critical order $\alpha=(N-1)/2$ play an essential role also in other questions about Fourier series of functions in several variables.

III. CONCLUSION

There is a continuous function in two variables with a Fourier series that does not converge over rectangles at any interior point of the square $[0,2\pi]^N$ (see [Fe]).

Certain results about Fourier series in the trigonometric system can be generalized considerably; for example, they can be carried over in a corresponding way to the spectral decompositions corresponding to self-adjoint elliptic differential operators.

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