Jacobi and Gauss-Seidel Iterative Methods for the Solution of Systems of Linear Equations Comparison

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***Abstract*- In our review paper we have compared the two iterative\ methods of solving system of linear equation, these iterative methods are used for solving sparse and dense system of linear equation. The methods being considered here are: Jacobi method and Gauss-Seidel method. Then the results give us the proof that Gauss-Seidel method is more efficient than Jacobi method by considering maximum number of iteration required to converge and higher accuracy.**

INTRODUCTION

The evolution of numerical methods on a daily basis is to discover the right solution techniques for solving problems in the field of applied science as well as pure science for example: weather forecasting, population analysis, studying the spread of a disease, predicting chemical reactions, physics, optics, etc.

Various problems in applied mathematics demand the solving systems of linear equations, with the linear system occurring naturally in some cases and as a part of the solution process in other cases. The collections of linear equations are called linear systems of equations. They involve same set of variables. Numerous methods have been

Introduced to solve systems of linear equations but there is no single method that is best forms all situations. These methods should be determined according to speed and accuracy. Also, speed is an indispensable factor in solving large systems of equations because the volume of computations involved is huge.

It is seen that most of the systems of linear equations come up in a large number of areas both directly in modeling physical situations and indirectly in the numerical solutions of the other mathematical models. These applications can be seen occurring in virtually all areas of the physical, biological and social science. A linear equation in the variable Y1,Y2, Y3……Yn is any equation of the form

a1y1+a2y2 +...+anyn= b1

There exist numerous approaches of solving system of linear equations i.e. direct methods and indirect (iterative) methods. The direct methods always provides us with the precise solution for the problem in which there is no error except the round off error due to the machine processes, whereas the iterative methods give us the approximate solutions in which there is usually some error.

The methods to solve linear systems of equations can be split into two halves: 1) Direct Methods and 2) Iterative Methods. Direct methods are not appropriate for solving large number of equations in a system, particularly when the coefficient matrix is sparse. Example faced problem with Gauss Elimination approach because of round off errors and slow convergence for large systems of equations. Iterative methods are highly effective as far as computer storage and time requirements are concerned.

JACOBI METHOD

This is the very first iterative technique, called the Jacobi method, named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851), Jacobi was one of the famous algorists to formulate an iterative method of solving system of linear equations. This particular method makes two assumptions:

1. The system given has a unique solution and
2. The coefficient matrix A does not contain any zeros on its core diagonal.

If any of the diagonal entries are zero, then rows or columns must be interchanged to avail a coefficient matrix that has non-zero entries on the main diagonal.

GAUSS-SEIDEL METHOD

This method is nothing but a slightly modified version of the Jacobi method. This method is called Gauss-Seidel method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is as easy to use as the Jacobi method, and it often takes fewer iterations to produce the same degree of accuracy. With the Jacobi method, the values of obtained in the n^{th} approximation remain unchanged until the entire n^{th} approximation has been computed. On the other hand, with the Gauss- Seidel method, we employ the new values of each as soon as they are known. That is, once we have found from the first equation, its value is then used in the second equation to obtain the new values. Likewise, the new value and the first value are used in the third equation to obtain the new and so on.

Method – 1 Analysis:-

The Jacobi method was obtained by solving the i^{th} equation in Ax = b, to obtain x_i (Provided $a_{ii} \neq 0$) i. e. given a system of linear equation

$a_{11}x_1 + a_{12}x_2+a_{13}x_3+\ldots\ldots\ldots\ldots a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2+a_{23}x_3+\ldots\ldots\ldots\ldots a_{2n}x_n = b_2$

$a_{31}x_1 + a_{32}x_2+a_{33}x_3+\ldots\ldots\ldots\ldots a_{3n}x_n = b_3$

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$a_nx_1 + a_{n2}x_2+a_{n3}x_3+\ldots\ldots\ldots\ldots a_{nn}x_n = b_n$

To start, solve the first equation for x_1, second equation for x_2, third equation for x_3 and so on… to obtain

$$X_1^{k+1} = \frac{1}{a11}(b_1 - a_{12}x_2^{k} - a_{13}x_3^{k} - \ldots\ldots - a_{1n}x_n^{k})$$

$$X_2^{k+1} = \frac{1}{a22}(b_2 - a_{21}x_1^{k} - a_{23}x_3^{k} - \ldots\ldots - a_{2n}x_n^{k})$$

$$X_3^{k+1} = \frac{1}{a33}(b_3 - a_{31}x_1^{k} - a_{32}x_2^{k} - \ldots\ldots - a_{3n}x_n^{k})$$

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$$X_1^{k+1} = \frac{1}{ann}(b_n - a_{n1}x_1^{k} - a_{n2}x_2^{k} - \ldots\ldots - a_{nn-1}\, x^{k}_{n-1})$$

Then make the initial guess ($zero^{th}$ iteration) for the solution

$x(0) = (x_1 (0), x_2(0),x_3 (0). . .x_n (0))$

substitute these values into the right hand side of (3). This constitute first iteration

$x^1 = (x_1 (1),x_2 (1),x_3 (1) . . .x_n (1))$.

Second iteration is obtained by substituting first iteration into the left hand side of (3.2), that is

$x^2 = (x_1 (2),x_2 (2),x_3 (2) . . .x_n (2))$.

And so on. The Jacobi method can be generalize as for each $k \geq 0$ we can generate the component $x^{k+1}{}_i$ of x^{k+1} from x^k by

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n}(-a_{ij}x_j^{k}) + b1\right]$$

For i = 1,2,3……. n

The Jacobi method in matrix form can be found by considering an n×n system of linear equation Ax=b where

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix},\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}.$$

We split matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ 0 & 0 & a_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ -a_{21} & 0 & \ldots & 0 & 0 \\ -a_{31} & -a_{32} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & -a_{n,n-1} & 0 \end{pmatrix} -$$

$$\begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \ldots & -a_{1n} \\ 0 & -a_{22} & -a_{23} & \ldots & -a_{2n} \\ 0 & 0 & 0 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \quad = D - L - U.$$

Therefore the matrix Ax =b can be transformed into (D – L – U) x = b, this implies that

Dx = (L +U)x + b.

METHOD-2 ANALYSIS

With Jacobi method, the value of x^{k+1} was obtained in $(k+1)^{th}$ iteration remain unchanged until the entire (k+1)th iteration has been calculated. With Gauss-Seidel method we use the value of x_i^{k+1} as soon as they are known. That is

$$x_1^{k+1} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k)$$

$$x_2^{k+1} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - \ldots - a_{2n}x_n^k)$$

$$x_3^{k+1} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - \ldots - a_{3n}x_n^k)$$

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$$x_n^{k+1} = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{k+1} - a_{n2}x_2^{k+1} - \ldots - a_{nn-1}x_{n-1}^{k+1})$$

Then make the initial guess (zero iteration) for the solution x (0) = (x_1 (0), x_2(0), x_3 (0). . .x_n (0)) substitute these value into the right of (3.3). This constitutes first iteration, x^1 = (x_1 (1), x_2 (1), x_3 (1) . . . x_n (1)). Second iteration is obtained by substituting first iteration into the left hand side of (3.3), x^2 = (x_1 (2), x_2 (2), x_3 (2) . . .x_n (2)).

And so on. This method can be generalize as for each k ≥ 0 we can generate the component x_i^{k+1} of x^{k+1} from x^k by

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i=1}\left(a_{ij}x_j^k\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{k+1}\right) + b1\right]$$

For i=1, 2...n

The Gauss-Seidel method in a matrix form is given by

$$(D - L)x^{k+1} = Ux^k + b$$
$$x^{k+1} = (D - L)^{-1}Ux^k + (D - L)^{-1}b$$

ITERATIVE METHOD'S CONVERGENCE

How fast the error | − | goes to zero as k, the number of iteration increases is determined by the rate of convergence of iterative methods, the sufficient condition of convergence for iterative methods define as

$X^{k+1} = Bx^k + C$ tp converge is that

Where ρ (B) = $\max_{1<i<n}|\lambda_i(B)| < 1$ where ρ (B) is the spectral radius of B.

This condition is fulfilled for both Jacobi and Gauss-Seidel methods if the coefficient matrix is diagonally dominant for any choice of initial approximation.

THEOREM 1: For any x^0 in R^n the sequence{ x^{k+1}} where k= 0 to ∞ dedfined by $x^{k+1}= Tx^k + c$, for each K≥0 converge to a unique solution of x = Tx + c iffρ(T) < 1

Absolute error = |True value – approximate value|

Relative error: absolute error ÷|True value|

Percentage relative error =Relative error × 100%

NUMERICAL EXPERIMENTS

The following problem in system of linear equations shall be considered:

PROBLEM 1

Solve the equation using Jacobi's method.

$$4x_1 - x_2 - x_4 = 0$$
$$-x_1 + 4x_2 - x_3 - x_5 = 5$$
$$-x_2 + 4x_3 - x_6 = 0$$
$$-x_1 + x_4 - x_5 = 6$$
$$-x_2 - x_4 + 4x_5 - x_6 = -2$$
$$-x_3 - x_5 + 4x_6 = 6$$

Taking the initial approximation which is $x_1^k = x_2^k = x_3^k = x_4^k = x_5^k = x_6^k = 0$ starting with these values and continuous to iterate we obtain the solution in the table below:

Table 1. Iteration result for Jacobi method

Iteration	X_1	X_2	X_3	X_4	X_5	X_6
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	1.250000	0.000000	1.500000	-0.50000	1.500000
2	0.687500	1.125000	0.687500	1.375000	0.562500	1.375000
3	0.625000	1.73445	0.625000	1.812575	0.46875	1.812575
4	0.886756	1.679688	0.886756	1.773438	0.839900	1.773438
5	0.823282	1.898554	0.823282	1.926865	0.806641	1.926865
6	0.956355	1.883301	0.956355	1.917481	0.938668	1.917481
7	0.950196	1.962695	0.950196	1.973606	0.929566	1.973606
8	0.984075	1.957490	0.984075	1.969941	0.977477	1.969941
9	0.981858	1.986497	0.981858	1.990388	0.974343	1.990388
10	0.994199	1.984515	0.994199	1.989050	0.991796	1.989050
11	0.993391	1.995041	0.993391	1.996499	0.990684	1.996499
12	0.997887	1.994354	0.997887	1.996011	0.997011	1.996011
13	0.997593	1.99816	0.997593	1.998725	0.996595	1.998728
14	0.999230	1.997945	0.999230	1.998547	0.998912	1.998547
15	0.999123	1.999343	0.999123	1.999536	0.998760	1.999536
16	0.999720	1.999252	0.999720	1.999471	0.999604	1.999471
17	0.999681	1.999761	0.999681	1.999831	0.999549	1.999831
18	0.999898	1.999728	0.999898	1.999808	0.999856	1.999808

This complete the table of the solution to the system of linear equation given above and the values of x_i are

(x_1,x_2,x_3,x_4,x_5,x_6) = (0.999898, 1.999728, 0.999898, 1.999808, 0.999856, 1.999808) respectively.

PROBLEM 2

Solve the equation using Gauss-Seidel method.

$$4x_1 - x_2 - x_4 = 0$$
$$-x_1 + 4x_2 - x_3 - x_5 = 5$$
$$-x_2 + 4x_3 - x_6 = 0$$
$$-x_1 + x_4 - x_5 = 6$$
$$-x_2 - x_4 + 4x_5 - x_6 = -2$$
$$-x_3 - x_5 + 4x_6 = 6$$

Taking the initial approximation

$x_1^k = x_2^k = x_3^k = x_4^k = x_5^k = x_6^k = 0$

Table 2. Iteration Result for Gauss-Seidel Method

Iteration	X_1	X_2	X_3	X_4	X_5	X_6
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	1.250000	0.312500	1.500000	0.187500	1.625000
2	o.687500	1.546875	0.79296	1.718750	0.722656	1.878906
3	0.816406	1.833008	0.927979	1.884766	0.899188	1.956792
4	0.929444	1.939153	0.973986	1.957158	0.963276	1.984316
5	0.974078	1.977835	0.990538	1.984339	0.986623	1.994290
6	0.990544	1.991926	0.996554	1.994292	0.995127	1.997920
7	0.996555	1.997059	0.998745	1.997921	0.998225	1.999243
8	0.998745	1.998929	0.999543	1.999243	0.999354	1.999724
9	0.999543	1.999610	0.999834	1.999724	0.999765	1.999900
10	0.999834	1.999858	0.999940	1.999900	0.999915	1.999964

Hence the solution is obtain after ten successive iteration we have x_1 = 0.999834, x_2= 1.999858, x_3=0.999940, x_4=1.999900, x_5=0.999915, x_6=1.999964.

ERROR ANALYSIS OF JACOBI METHOD

The true values are:
(X_1,X_2,X_3,X_4,X_5,X_6) = (1, 2, 1, 2, 1, 2)
while the computed values are (0.999898, 1.999728, 0.999898, 1.999808, 0.999856, 1.999808). Hence we determine the error as follows by using
Absolute error =|True value – approximate value|
=|2 - 1.999728| = 0.000272
Percentage relative error = 0.000136*100% =0.0136%

ERROR ANALYSIS OF GAUSS-SEIDEL METHOD

The true value are (X_1,X_2,X_3,X_4,X_5,X_6) = (1, 2, 1, 2, 1, 2) while the computed value are, (0.999834, 1.999858, 0.999940, 1.999900, 0.999915, 1.999964).
Hence we determine the error as follows by using
Absolute error =|True value – approximate value| =|2 - 1.999834|=0.000166
Percentage relative error = 0.000166*100% =0.0083%

CONTRAST OF TWO ITERATIVE TECHNIQUES USED IN THE EVALUATION

As we know, the current values of the unknowns at each stage of the iteration are used in proceeding to another stage of iteration. Here, the numerical results and errors analysis of these two iterative methods for the system of linear equations showed that second method is more rapid in convergence as compared to that of the first method.

CONCLUSION

Tablei3.Showing percentage error of both methods

Methods	No of iteration	Error%
Jacobi	18	0.0136
Gauss-Seidel	10	0.0083

There exist various methods of solving system of linear equations I, some are direct methods while some are numerical method. In this review paper, two iterative methods of solving system of linear equations have been showed where the Gauss Seidel method had proved to be the most precise and effective in the sense that it converges very fast. From the practical example, I we also observe that the necessary solution was obtained with very little iterations easily without much problem relating to the starting condition.

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