Fluid Flows in Forced Convection

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Abstract- **The fluid flow equations of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant wall temperature are derived and solutions of these equations are investigated using DTM-Pade´ approximation. Velocity and temperature profiles are plotted.**

Index terms- **Forced convection, Thermal Boundary Layer, DTM-Pade approximation**

1. FORCED CONVECTION

Consider the steady flow of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant temperature Tw placed along the direction of a uniform stream of velocity U∞ and temperature T∞[1]. Let the origin of co-ordinates be at the leading edge of the plate, the x-axis along the plate and y-axis normal to it. Equations for two dimensional, laminar, steady boundary layer flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=U\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$\rho C_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=\kappa\frac{\partial^2 y}{\partial x^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

In the present case, $U(x) = U_\infty$ (constant) Thus (1) reduces to,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=v\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 \qquad (4)$$

With the boundary conditions,

$y=0; u=v=0, T=T_w$ (*Isothermal*)

$\frac{\partial T}{\partial y}=0$ (Adiabatic) (5)

$y=\infty; u=u_\infty, T=T_\infty$

2. Integrals of The Thermal Boundary Layer Equation with Prandtl Number of The Fluid(Pr=1)

A simple integral of the equation (4) can be obtained immediately if the frictional heat is neglected and the Prandtl number of the fluid is unity.

In such cases the two equations are

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=v\frac{\partial^2 u}{\partial y^2} \qquad (6)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2} \qquad (7)$$

With the boundary conditions

$y=\infty; u=U_\infty, T=T_\infty$ (8)

When $\gamma=a$ thus $P_r=\frac{\gamma}{a}=1$ (7) becomes identical to (6) with boundary conditions if $\frac{T-T_W}{T_\infty-T_W}$ is replaced by $\frac{u}{U_\infty}$

Hence, $\frac{T-T_W}{T_\infty-T_W}=\frac{u}{U_\infty}$ or (9)

$$\frac{T-T_\infty}{T_w-T_\infty}=1-\frac{u}{U_\infty}\ (P_r=1)$$

It is known as Crocco's first integral. We can also write this as

$$\frac{\partial}{\partial y}\left(\frac{u}{U_\infty}\right)=\frac{\partial}{\partial y}\left(\frac{T-T_W}{T_\infty-T_W}\right) \qquad (10)$$

This shows that the heat-flux and the skin-friction are proportional to each other. To get the exact relationship between the two, we write the value of the local Nusselt number, which is given by

$$Nu(x)=-\frac{x\left(\frac{\partial T}{\partial y}\right)_0}{T_W-T_\infty}=\frac{x}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_0 \qquad (11)$$

$$Nu(x) = \frac{x}{\mu U_\infty} T_w = \frac{x}{\mu U_\infty}\left(\frac{1}{2}\rho {U_\infty}^2 C_f\right) \qquad (12)$$

Thus $Nu(x) = \frac{1}{2} Re_x C_f$

Where $Re_x = \frac{U_\infty x}{\gamma}$ and $\gamma = \frac{\mu}{\rho}$

This is known as Reynold's Analogy. If the frictional heat is not neglected but the wall is insulated then another simple integral of (4) is possible again when Pr=1. We have $T = T(u)$ then (4) can be written as

$$u\frac{dT}{du}\frac{\partial u}{\partial x} + v\frac{dT}{du}\frac{\partial u}{\partial y} = a\frac{\partial}{\partial y}\left[\frac{dT}{du}\frac{\partial u}{\partial y}\right] + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2$$

$$(\gamma - a)T_u\frac{\partial^2 u}{\partial y^2} = \left[aT_{uu} + \frac{\mu}{\rho C_p}\right]\left(\frac{\partial u}{\partial y}\right)^2 \qquad (13)$$

(13) will be identically satisfied. Thus $T = T(u)$ will be the solution of (4) if

$\gamma = a$ and $T_{uu} = -\frac{\mu}{a\rho C_p}$

$Pr = 1$ and $T_{uu} = -\frac{\gamma}{aC_p}\ \left(since\ \gamma = \frac{\mu}{\rho}\right)$

$$T_{uu} = -\frac{1}{C_p}\ (since\ \gamma = a)$$

where the constants of integration vanishes, since at $y = 0: u = 0, \frac{\partial u}{\partial y} \neq 0\ and\ \frac{\partial T}{\partial y} = 0$

That implies$T_u = 0$. Integrating and using the boundary condition at infinity That is $u = U_\infty$ and $T = T_\infty$, we obtain $T_u = -\frac{u}{C_p}$

$$[T]^{T}_{T_\infty} = -\left[\frac{u^2}{2C_p}\right]^{u}_{U_\infty}$$

$$T - T_\infty = -\frac{1}{2C_p}\left[u^2 - {U_\infty}^2\right] + constant$$

$$T - T_\infty = -\frac{u^2}{2C_p} + \frac{{U_\infty}^2}{2C_p}$$

$$T - T_\infty = \frac{{U_\infty}^2}{2C_p}\left[1 - \frac{u^2}{{U_\infty}^2}\right]$$

$$\frac{T - T_\infty}{\frac{{U_\infty}^2}{2C_p}} = 1 - \left(\frac{u^2}{{U_\infty}^2}\right)\ since\ Pr = 1 \qquad (14)$$

This is known as Crocco's second integral.

3. Integrals of the Thermal Boundary Layer Equation for Arbitrary Values of the Prandtl Number(Pr)

For the solution of (4), we shall require the velocity distribution which was obtained by Blasius and is as follows

$u = U_\infty \emptyset'(n)$

$$v = \frac{1}{2}\sqrt{\frac{\gamma U_\infty}{x}}\left(n\emptyset'(n) - \emptyset(n)\right) \ (15)$$

Where $\eta = y\sqrt{\frac{U_\infty}{\gamma x}}$ (Similarity variable)

$$\frac{\partial u}{\partial x} = U_\infty \emptyset''(\eta)\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial x} = U_\infty \emptyset''(\eta)\left[y\sqrt{\frac{U_\infty}{\gamma}}\left(-\frac{1}{2}\right)x^{\left(-\frac{3}{2}\right)}\right]$$

$$\frac{\partial u}{\partial x} = -U_\infty\sqrt{\frac{U_\infty}{\gamma x}}\frac{y}{2x}\emptyset''(\eta)$$

$$\frac{\partial u}{\partial x} = U_\infty \emptyset''(\eta)\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \emptyset'''(\eta)\sqrt{\frac{U_\infty}{\gamma x}}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \emptyset'''(\eta)\sqrt{\frac{U_\infty}{\gamma x}}\sqrt{\frac{U_\infty}{\gamma x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{{U_\infty}^2}{\gamma x}\emptyset'''(\eta)$$

We have,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \gamma\frac{\partial^2 u}{\partial y^2}$$

$$-U_\infty \emptyset'(n) U_\infty\sqrt{\frac{U_\infty}{\gamma x}}\frac{y}{2x}\emptyset''(\eta) + U_\infty\sqrt{\frac{U_\infty}{\gamma x}}\emptyset''(\eta)\frac{1}{2}\sqrt{\frac{\gamma U_\infty}{x}}\left(n\emptyset'(n) - \emptyset(n)\right) = \frac{\gamma {U_\infty}^2}{\gamma x}\emptyset'''(\eta)$$

$$\frac{U_\infty{}^2}{x}\emptyset'''(\eta) + \frac{U_\infty{}^2}{2x}\emptyset''(\eta)\emptyset(\eta) = 0$$

$$2\emptyset''' + \emptyset\emptyset'' = 0 \quad (16)$$

The function $\emptyset(\eta)$ satisfies the above differential equation.
With the boundary conditions
$\eta = 0;\ \emptyset = \emptyset' = 0;$
$\eta = \infty;\ \emptyset' = 1$ (17)
In order to obtain a complete integral of the (4) for an isothermal wall, it will be easier to calculate first the solution of it when the dissipation term is neglected. i.e., the solution of the cooling problem with a prescribed value of (T_w-T_∞) and then another solution when the frictional heat is accounted but the wall is adiabatic. i.e.,the problem of plate thermometer. Since the (4) is a linear differential equation, to get the complete integral, the two solutions may then be properly superimposed.

4. SOLUTION OF THE COOLING PROBLEM

Consider (7),

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2} \quad (18)$$

With the boundary conditions
$y = 0; T = T_w$ and
$y = \infty : T = T_\infty$ (19)

We have

$$\theta_1 = \frac{T - T_\infty}{T_w - T_\infty}$$

$$T - T_\infty = \theta_1(\eta)T_w - T_\infty \quad (20)$$

$$T = T_\infty + \theta_1(\eta)T_w - T_\infty$$

Solution of (18) in which θ_1 is a solution of the similarly variable η only or in other words we look for a similar solution of θ_1.
Differentiate (20) w.r.t. x

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\emptyset'(\eta)\frac{\partial \eta}{\partial x}$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\emptyset'(\eta)\left[\sqrt{\frac{U_\infty}{\gamma}}\left(-\frac{y}{2x}\right)\right] \quad (21)$$

Differentiating (20) w.r.t. y

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\emptyset'(\eta)\frac{\partial \eta}{\partial y}$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\emptyset'(\eta)\left[\sqrt{\frac{U_\infty}{\gamma x}}\right] \quad (22)$$

Differentiate (22) w.r.t. y

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\emptyset'(\eta)\frac{U_\infty}{\gamma x} \quad (23)$$

Substituting the values of (15),(21),(22),(23) in (18) we get

$$\begin{aligned} U_\infty(T_w - T_\infty)\emptyset'(\eta)\left(-\frac{y}{2x}\right)\sqrt{\frac{U_\infty}{\gamma x}} \\ + \frac{1}{2}\sqrt{\frac{\gamma U_\infty}{x}}[\eta\emptyset'(\eta) - \emptyset(\eta)](T_w \\ - T_\infty)\emptyset'(\eta)\left[\sqrt{\frac{U_\infty}{\gamma x}}\right] \\ = a(T_w - T_\infty)\emptyset''(\eta)\frac{U_\infty}{\gamma x} \end{aligned}$$

Dividing throughout by $\frac{aU_\infty}{\gamma x} - \emptyset(\eta)\theta_1{}'(\eta)\frac{\gamma}{2a} = \theta_1{}''(\eta)$

$$\theta_1{}''(\eta) = -\frac{1}{2}\emptyset(\eta)\theta_1{}'(\eta)\frac{\gamma}{2a}$$

But $\frac{\gamma}{a} = Pr$

$$\theta_1{}'' + \frac{Pr}{2}\emptyset\theta_1{}' = 0 \quad (24)$$

With the boundary conditions
$\eta = 0\,; \theta_1 = 1\ and\ \eta = \infty\,; \theta_1 = 0$ (25)
From (24) we have,

$$\theta_1{}'' + \frac{Pr}{2}\emptyset\theta_1{}' = 0$$

$$\frac{\theta_1{}''}{\theta_1} = -\frac{1}{2}\,Pr\emptyset$$

$$\theta_1{}'' + \frac{Pr}{2}\emptyset\theta_1{}' = 0$$

$$\frac{\theta_1{}''}{\theta_1} = -\frac{1}{2}\,Pr\emptyset\left[\frac{-2\emptyset'''}{\emptyset''}\right]$$

[since from (16)]

$$\frac{{\theta_1}''}{\theta_1} = Pr\left[\frac{\emptyset'''}{\emptyset''}\right] \qquad (26)$$

Integrating w.r.t. ɳ we get,

$$\frac{{\theta_1}''}{\theta_1} = Pr\left[\frac{\emptyset''}{\emptyset'}\right]$$

Integrating once again w.r.t. ɳ we get,

$$\log {\theta_1}' = Pr \log \emptyset'' + \log A$$

Where A is the constant of integration

$$\log {\theta_1}' = \log {\emptyset''}^{Pr} + \log A$$

$$\log\left(\frac{{\theta_1}'}{A}\right) = \log {\emptyset''}^{Pr}$$

$$\theta_1 = \frac{[\emptyset'(ɳ)]^{\infty}{}_{ɳ}}{[\emptyset'(ɳ)]^{\infty}{}_{0}}$$

$$\frac{{\theta_1}'}{A} = [\emptyset(ɳ)]^{Pr}$$

$${\theta_1}' = A[\emptyset''(ɳ)]^{Pr} \qquad (27)$$

Integrating (27) w.r,t. ɳ from ∞ to ɳ we get

$$\theta_1(ɳ, \mathrm{Pr}) = A\int_{\infty}^{ɳ}[\emptyset''(ɳ)]^{Pr}\, dɳ$$

$$A = \frac{-\theta_1(ɳ, Pr)}{\int_{\infty}^{ɳ}[\emptyset''(ɳ)]^{Pr}\, dɳ} \qquad (28)$$

Using the first boundary condition, i.e $ɳ = 0\,; \theta_1 = 1$

$$A = \frac{-1}{\int_0^{\infty}[\emptyset''(ɳ)]^{Pr}\, dɳ} \qquad (29)$$

Equating (28) and (29), we get

$$\frac{-\theta_1(ɳ, Pr)}{\int_{\infty}^{ɳ}[\emptyset''(ɳ)]^{Pr}\, dɳ} = \frac{-1}{\int_0^{\infty}[\emptyset''(ɳ)]^{Pr}\, dɳ}$$

$$\theta_1(ɳ, Pr) = \frac{\int_{\infty}^{ɳ}[\emptyset''(ɳ)]^{Pr}\, dɳ}{\int_0^{\infty}[\emptyset''(ɳ)]^{Pr}\, dɳ}$$

When Prandtl number (Pr) of the fluid is unity, the above equation becomes

5 RESULTS AND DISCUSSIONS

The velocity profiles for Pr=1 and temperature profile for varying Prandtl number is plotted in graphs 1 and 2 respectively.

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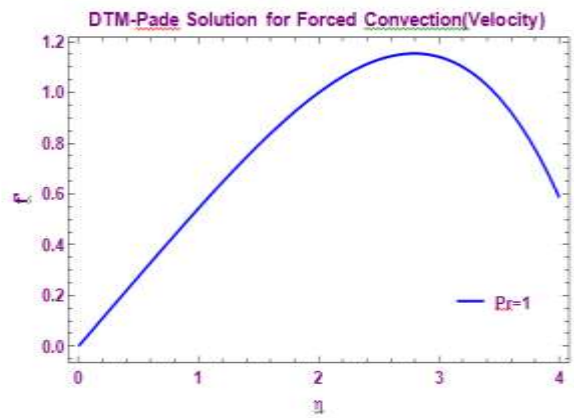

Figure 1: Velocity Profile in the laminar boundary layer with Pr=1

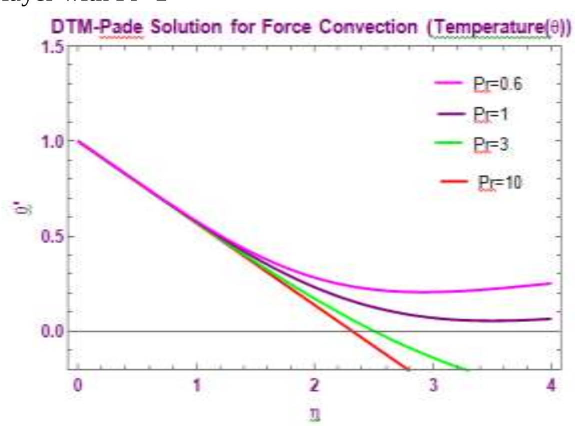

Figure 2: Temperature Distribution in the laminar boundary Layer for different Prandtl Numbers.

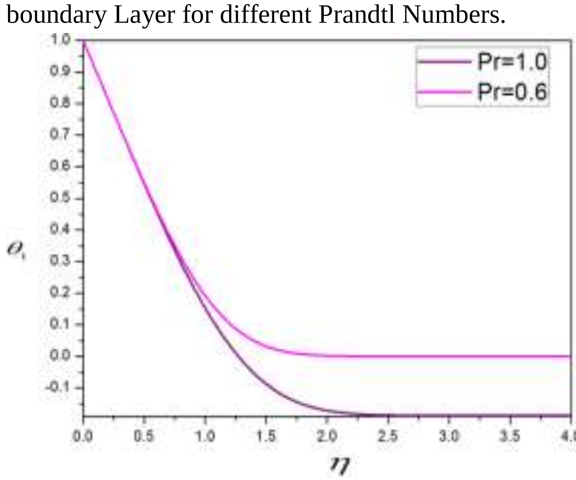

Figure 3: Numerical Solution using Runge-kutta 4th order.

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