Fixed Point Theorem for Mappings in Complex Valued Metric Space

Rafia Aziz

Government Girls College, BTI Road, Anand Nagar, Khargone, Madhya Pradesh 451001 India

***Abstract* - In [1] Azam et al introduced the notion of complex valued metric spaces and obtained common fixed-point result for mappings in the context of complex valued metric spaces. In this paper, existence of common fixed point is established for two mappings satisfying certain rational inequality on complex valued metric space followed by the consequences of the main result. These results complement the comparable results from the current literature.**

***Index Terms* - Common fixed point, Cauchy sequence, Complex valued metric space.**

Mathematics Subject Classification: 47H10, 54H25, 46J10, 55M20.

1.INTRODUCTION

The concept of complex valued metric space was introduced by Azam et al. [1], proving some fixed-point results for mappings satisfying a rational inequality in complex valued metric spaces. Afterwards, several papers have dealt with fixed point theory in complex valued metric spaces (see [2], [3], [4] and references therein). In [5], Saluja proved some fixed-point theorems under rational contraction in the setting of complex valued metric spaces. Motivated by these results, in this paper, we establish the existence and uniqueness of common fixed points for a pair of self-mappings satisfying a generalized rational contractive condition in complex valued metric spaces.

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion. Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\leq$ on C as follows: $z_1 \leq z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$

Consequently, one can infer that $z_1 \leq z_2$ if one of the following conditions is satisfied:

$(i)\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$ $\quad (ii)\ \mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

$(iii)\ \mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$ $\quad (iv)\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

In particular, we write $z_1 \leq z_2$ if one of (i), (ii) and (iii) is satisfied and we write $z_1 < z_2$ if only (iii) is satisfied.

Definition 1.1 Let X be a nonempty set whereas C be the set of complex numbers. Suppose that the mapping $d : X \times X \to C$, satisfies the following conditions

(i) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$

Then *d* is called a complex valued metric on X and (*X*, *d*) is called complex valued metric space.

Definition 1.2 Let (*X*, *d*) be a complex valued metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 < r \in C$ such that $N(b, r) \subseteq B$ where

$$N(b, r) = \{y \in X : d(b, y) < r\}$$

(ii) A point $x \in X$ is called a limit point of B whenever for every

$$0 < r \in C, N(x, r) \cap (B \setminus \{X\}) \neq \phi$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A whereas a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$F=\{N(x,r): x\in X, 0<r\}$ is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdroff.

Definition 1.3 Let (*X*, *d*) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x\in X$. We say that

(i) the sequence $\{x_n\}_{n\geq 1}$ converges to *x* if for every $c\in C$ with $c>0$ there is $n_0\in N$ such that for all $n>n_0, d(x_n,x)<c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n\to x,$ as $n\to\infty$

(ii) the sequence $\{x_n\}_{n\geq 1}$ is Cauchy's sequence if for every $c\in C$ with $c>0$ there is $n_0\in N$ such that for all $n>n_0, d(x_n,x_{n+m})<c$,

(iii) the metric space (*X*, *d*) is a complete complex valued metric space if every Cauchy's sequence is convergent.

In [1], Azam et al. established the following two lemmas:

Lemma 1.4. Let (*X*, *d*) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to *x* if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.

Lemma 1.5. Let (*X*, *d*) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy's sequence if and only if $|d(x_n,x_{n+m})|\to 0$ as $n\to\infty$.

2. MAIN RESULTS

Theorem 2.1. Let (*X*, *d*) be a complete complex valued metric space. Let the mappings $S,T: X\to X$ satisfy.

$$d(Sx,Ty)\leq \alpha\frac{\left[1+d^2(x,Sx)\right]d(y,Ty)}{[1+d^2(x,y)]}+\beta\frac{\left[1+d(x,Ty)\right]d(y,Ty)}{\left[1+d(x,Sx)+d(y,Ty)\right]}+\gamma d(x,y) \quad ....(2.1)$$

for all *x*, *y* in *X*, where $\alpha+\beta+\gamma<1$, $\alpha+\beta<1, \gamma<1, \alpha>0, \beta>0, \gamma>0$. Then S and T have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $x_{n+1}=Sx_n$ and $x_{n+2}=Tx_{n+1}$ *n* = 0,1, 2, 3, … . Then by (2.1)

$$d(x_{n+1},x_{n+2})=d(Sx_n,Tx_{n+1})$$

$$\leq \alpha\frac{\left[1+d^2(x_n,Sx_n)\right]d(x_{n+1},Tx_{n+1})}{[1+d^2(x_n,x_{n+1})]}+\beta\frac{\left[1+d(x_n,Tx_{n+1})\right]d(x_{n+1},Tx_{n+1})}{\left[1+d(x_n,Sx_n)+d(x_{n+1},Tx_{n+1})\right]}+\gamma d(x_n,x_{n+1})$$

$$\leq \alpha\frac{\left[1+d^2(x_n,x_{n+1})\right]d(x_{n+1},x_{n+2})}{[1+d^2(x_n,x_{n+1})]}+\beta\frac{\left[1+d(x_n,x_{n+2})\right]d(x_{n+1},x_{n+2})}{\left[1+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})\right]}+\gamma d(x_n,x_{n+1})$$

$$\leq \alpha d(x_{n+1},x_{n+2})+\beta\frac{\left[1+d(x_n,x_{n+2})\right]d(x_{n+1},x_{n+2})}{\left[1+d(x_n,x_{n+2})\right]}+\gamma d(x_n,x_{n+1})$$

$$\leq \alpha d(x_{n+1},x_{n+2})+\beta d(x_{n+1},x_{n+2})+\gamma d(x_n,x_{n+1})$$

$$\leq \alpha d(x_{n+1},x_{n+2})+\beta d(x_{n+1},x_{n+2})+\gamma d(x_n,x_{n+1})$$

This implies that

$$d(x_{n+1},x_{n+2})-\alpha d(x_{n+1},x_{n+2})-\beta d(x_{n+1},x_{n+2})\le \gamma d(x_n,x_{n+1})$$

$$\text{or}\quad d(x_{n+1},x_{n+2})(1-\alpha-\beta)\le \gamma d(x_n,x_{n+1})$$

$$\text{or}\quad d(x_{n+1},x_{n+2})\le \frac{\gamma}{(1-\alpha-\beta)}d(x_n,x_{n+1})$$

$$\text{or}\quad d(x_{n+1},x_{n+2})\le \lambda d(x_n,x_{n+1})$$

for all $n\ge 0$, where $\lambda=\frac{\gamma}{(1-\alpha-\beta)}$ and $0<\lambda<1$ in view of the conditions

$$\alpha+\beta+\gamma<1,\ \alpha+\beta<1,\ \alpha>0,\ \beta>0,\gamma>0$$

Consequently $d(x_{n+1},x_{n+2})\le \lambda d(x_n,x_{n+1})\le ....\le \lambda^{n+1}d(x_0,x_1)$

Hence for all $m>n$

$$d(x_n,x_m)\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...............+d(x_{m-1},x_m)$$

$$\le(\lambda^n+\lambda^{n+1}+..............+\lambda^{m-1})d(x_0,x_1)$$

$$\le\left(\frac{\lambda^n}{1-\lambda}\right)d(x_0,x_1)$$

This implies that $|d(x_n,x_m)|\le\left(\frac{\lambda^n}{1-\lambda}\right)|d(x_0,x_1)|$

As $m,n\to\infty,|d(x_n,x_m)|\le\left(\frac{\lambda^n}{1-\lambda}\right)|d(x_0,x_1)|\to 0$, which amounts to saying that $\{x_n\}$ is a Cauchy's sequence in X which is Complete, so it converges to a point say *u* in X.

It remains to show that $Su=u$. Suppose this is not true and $d(u,Su)=z>0$. Then from (2.1) we have

$$z\le d(u,x_{n+2})+d(x_{n+2},Su)$$

$$\le d(u,x_{n+2})+d(Su,Tx_{n+1})$$

$$\le d(u,x_{n+2})+\alpha\frac{\left[1+d^2(u,Su)\right]d(x_{n+1},Tx_{n+1})}{[1+d^2(u,x_{n+1})]}+\beta\frac{\left[1+d(u,Tx_{n+1})\right]d(x_{n+1},Tx_{n+1})}{\left[1+d(u,Su)+d(x_{n+1},Tx_{n+1})\right]}+\gamma d(u,x_{n+1})$$

$$\le d(u,x_{n+2})+\alpha\frac{\left[1+d^2(u,Su)\right]d(x_{n+1},x_{n+2})}{[1+d^2(u,x_{n+1})]}+\beta\frac{\left[1+d(u,x_{n+2})\right]d(x_{n+1},x_{n+2})}{\left[1+d(u,Su)+d(x_{n+1},x_{n+2})\right]}+\gamma d(u,x_{n+1})$$

which implies that

$$|z|\le |d(u,x_{n+2})|+\alpha\frac{\left|\left[1+d^2(u,Su)\right]\right||d(x_{n+1},x_{n+2})|}{\left|[1+d^2(u,x_{n+1})]\right|}+\beta\frac{\left|[1+d(u,x_{n+2})]\right||d(x_{n+1},x_{n+2})|}{\left|[1+d(u,Su)+d(x_{n+1},x_{n+2})]\right|}+\gamma|d(u,x_{n+1})|$$

$$\le|d(u,x_{n+2})|+\alpha\frac{\left|\left[1+z^2\right]\right||d(x_{n+1},x_{n+2})|}{\left|[1+d^2(u,x_{n+1})]\right|}+\beta\frac{\left|[1+d(u,x_{n+2})]\right||d(x_{n+1},x_{n+2})|}{\left|[1+z+d(x_{n+1},x_{n+2})]\right|}+\gamma|d(u,x_{n+1})|$$

On taking limit $n \to \infty$, we obtain that $|z| \leq 0$ which is a contradiction. Hence $Su = u$. Similarly we can show $Tu = u$. Next we show that S and T have unique common fixed point. For this assume that u^* is another fixed point of S and T.

$$d(u,u^*) = d(Su,Tu^*)$$

$$\leq \alpha \frac{\left[1+d^2(u,Su)\right]d(u^*,Tu^*)}{[1+d^2(u,u^*)]} + \beta \frac{\left[1+d(u,Tu^*)\right]d(u^*,Tu^*)}{\left[1+d(u,Su)+d(u^*,Tu^*)\right]} + \gamma d(u,u^*)$$

$$\Rightarrow \quad \left|d(u,u^*)\right| \leq \alpha \frac{\left|\left[1+d^2(u,u)\right]\right|\left|d(u^*,u^*)\right|}{\left|[1+d^2(u,u^*)]\right|} + \beta \frac{\left|\left[1+d(u,u^*)\right]\right|\left|d(u^*,u^*)\right|}{\left|\left[1+d(u,u)+d(u^*,u^*)\right]\right|} + \gamma \left|d(u,u^*)\right|$$

$$\leq \gamma \left|d(u,u^*)\right|$$

This implies that $\left|d(u,u^*)\right|\left(1-\gamma\right) \leq 0$ or $\left|d(u,u^*)\right| \leq 0$ as $1-\gamma > 0$. Hence $u = u^*$. This proves the uniqueness of common fixed point.

Corollary 2.2. Let $\{T_k\}$ be a sequence of self-mappings defined on a complete complex valued metric space (*X*, *d*) satisfying

$$d(T_i x, T_j y) \leq \alpha \frac{\left[1+d^2(x,T_i x)\right]d(y,T_j y)}{[1+d^2(x,y)]} + \beta \frac{\left[1+d(x,T_j y)\right]d(y,T_j y)}{\left[1+d(x,T_i x)+d(y,T_j y)\right]} + \gamma d(x,y) \quad ....(2.2)$$

for all *x*, *y* in *X*, where $\alpha+\beta+\gamma<1$, $\alpha+\beta<1, \gamma<1, \alpha>0, \beta>0, \gamma>0$. Then $\{T_k\}$ have a unique common fixed point.

REFERENCES

[1] A. Azam, B. Fisher, M. Khan, Common fixed-point theorems in complex valued metric spaces, Numerical Functional Analysis and optimization, 32, No. 3 (2011), 243-253.

[2] F. Rouzkard and M. Imdad, Some common fixed-point theorems on complex valued metric spaces, Computer and Mathematics with Applications 64 (2012), 1866 – 1874.

[3] K. Sitthikul and S. Saejung, Some fixed-point theorems in complex valued metric spaces, Fixed Point Theory and Applications 2012 (2012), 189.

[4] W. Sintunavarat and P. Kumam, Generalized common fixed-point theorems in complex valued metric spaces and applications, Journal of Inequalities and Applications 2012(1) (2012), 84.

[5] G. S. Saluja, Fixed point theorems under rational contraction in complex valued metric spaces, Nonlinear Funct. Anal. Appl., 22(1) (2017) 209-216.

[6] S. Poonkundran, M. Dharmalingam, Common and coincidence Fixed point theorems under rational inequalities in complex valued metric spaces with Applications, International Journal of Computational and Applied Mathematics, 12(1) (2017) 241-254.

[7] W. Sintunavart, Y.J. Cho and P. Kumam, Urysohn integral equation approach by common fixed points in complex valued metric spaces, Advances in Difference Equations 2013 (2013), 49.

[8] A. Jamshaid, K. Chakkrid, A. Azam, Common fixed points for multivalued mappings in complex valued metric spaces with applications, Abstr. Appl. Anal. 2013 (2013) 12. (Article ID 854965)

[9] S. Bhatt, S. Chaukiyal, R.C. Dimri, A common fixed-point theorem for weakly compatible maps

in complex valued metric spaces, Int. J. Math. Sci. Appl. 1 (3) (2011) 1385–1389.

[10] S.M. Kang, M. Kumar, P. Kumar, S. Kumar, Coupled fixed point theorems in complex valued metric spaces, Int. J. Math. Anal. 7 (46) (2013) 2269–2277.

[11] S. Manro, some common fixed-point theorems in complex valued metric space using implicit function, Int. J. Anal. Appl. 2 (1) (2013) 62–70.

[12] M.A. Kutbi, A. Azam, A. Jamshaid, C. Bari, some coupled fixed points results for generalized contraction in complex valued metric spaces, J. Appl. Math. 2013 (2013). (Article ID 352927), 10 pages.

[13] H.K. Nashine, M. Imdad, M. Hassan, Common fixed-point theorems under rational contractions in complex valued metric spaces, J. Nonlinear Sci. Appl. 7 (2014) 42–50.

[14] R.K. Verma, H.K. Pathak, Common fixed point theorems using property (E.A.) in complex-valued metric spaces, Thai J. Math. 11 (2) (2013) 347–355.