On relativistic anisotropic compact stars

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***Abstract* - In the present article we provide a new model of compact star satisfying the Karmarkar condition. We proceed our calculations by assuming a new type of metric potential for g_{rr} and g_{tt} is obtained from the condition of embedding class one. The physical parameters are obtained by employing the metric potentials to the Einstein's field equations. Our model is free from central singularity and satisfies all the physical conditions. We have also investigated equilibrium and stability of compact star by several methods.**

***Index Terms* - Class one solution, Compact stars, General Relativity, Stability**

I.INTRODUCTION

This work is devoted to the study of spherically symmetric stellar models of cold stars using Karmarkar condition Karmarkar [1] and satisfying causality as well all energy conditions. Since the seminar paper by Oppenheimer and Snyder [2], most of the work dedicated to the problem of general relativistic gravitational collapse, deal with spherically symmetric fluid distribution. Exact solutions have been taking an important role in breaking the mystery of the universe. However, many physical equations sometimes cannot solve exactly. In the realm of general relativity, compact stars are the most fascinating objects we have ever observed. Their modeling can be done by exact solution method and numerical technique.

The simplest exact model can be found by using the Karmarkar condition. This condition allows that a 4-dimensional spacetime can be embedded in 5-dimensional flat-space called embedding class one. However, Pandey & Sharma [3] shows that to become a class one solution, Karmarkar condition alone is not sufficient but to satisfy $R_{2323} \neq 0$ as well.

The embedding problem was first considered by Schlai [4], who conjectured that a Riemannian manifold with positive defined and analytic metric can be locally and isometrically embedded as a sub- manifold of an Euclidean space E_N. The first global isometric embedding theorem of V_n, where n is the dimension of the Riemannian manifold, into E_N were established by Nash [5]. The relation between N and n is that $N = n(n+1)/2$. Recently, many researchers have been showing interest in the embedding problems to analyze the interior properties of compact stars via exact solutions of Einstein's field equations [6-9].

The paper is organized as follows: In the next section we introduce the notation, description of the fluid distribution and Einstein field equations. Section III is devoted to the finding of new exact solution. Section IV discussed the physical properties of the solution and Section V deals with boundary conditions. The stability and equilibrium of the solution are discuss Section VI. Finally, the conclusion is given in the last section.

II. THE FIELD EQUATIONS

The interior spacetime is assume to be spherically symmetric in canonical coordinate given as

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \; d\phi^2).$$

Assuming and anisotropic energy-momentum tensor of the form given below:

$$T_{\mu\nu} = \rho \, v_\mu v_\nu + p_r \chi_\mu \, \chi_\nu + p_t \left(v_\mu \, v_\nu - \chi_\mu \, \chi_\nu - g_{\mu\nu}\right),$$

where the symbols have their usual meanings. Now, the Einstein's field equations becomes

$$8\pi\rho = \frac{1 - e^{-\lambda}}{r^2} + \frac{\lambda' \, e^{-\lambda}}{r}$$

$$8\pi p_r = \frac{\nu' \, e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2}$$

$$8\pi p_t = \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \nu'\lambda' + \frac{2(\nu' - \lambda')}{r}\right),$$

where the primes denotes differentiation with respect to the radial coordinate r. The anisotropy in pressure is defined as $\Delta = 8\pi(p_t - p_r)$.

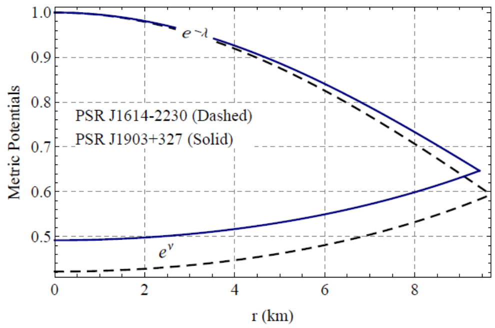

Fig 1. Variation of metric potentials.

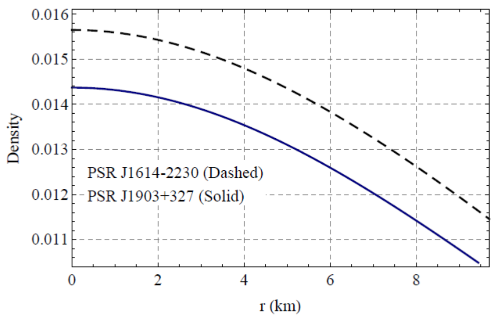

Fig 2. Variation of density.

To solve the field equations we are considering embedding class one spacetime i.e. the 4-dimensional spacetime can be embedded in 5-dimensional flat hyperspace. This is possible iff the 4-dimensional spacetime satisfies the Karmarkar condition i.e.

$$\frac{2\nu''}{\nu'}+\nu'=\frac{\lambda' e^{\lambda}}{e^{\lambda}-1}\,.$$

Using this condition the final expression for pressure anisotropy takes the form

$$\Delta=\frac{\nu'}{4e^{\lambda}}\left[\frac{2}{r}-\frac{\lambda'}{e^{\lambda}-1}\right]\left[\frac{\nu' e^{\nu}}{2rB^2}-1\right].$$

III. GENERATING NEW SOLUTION

Assuming a specific form of g_{rr} metric potential as

$$e^{\lambda}=1+ar^2\sec^2(c+br^2)$$

where a, b and c are arbitrary constants. Using the Karmarkar condition the other form of metric function becomes

$$e^{\nu}=\left[A+\frac{\sqrt{a}\,B}{2b}\ln\left\{\tan\left(\frac{\pi}{4}+\frac{c+br^2}{2}\right)\right\}\right]^2.$$

Here, A and B are constants of integration.

Now the field equations leads to the expressions of pressures, density and anisotropy as

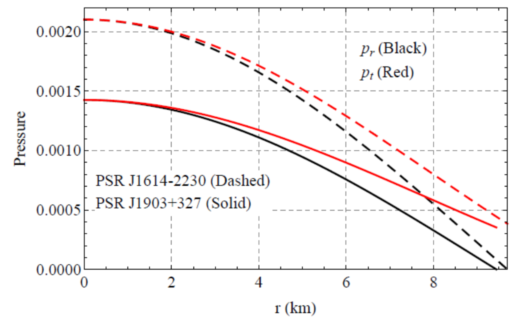

Fig 3. Variation of pressure.

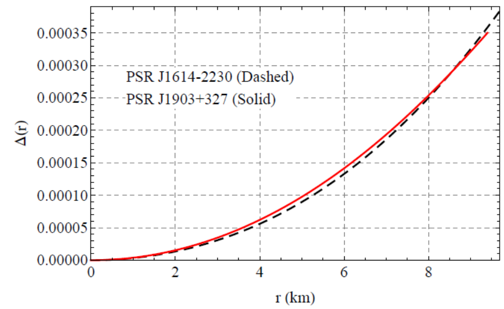

Fig 4. Variation of pressure anisotropy.

$$8\pi\rho=\frac{2a}{\{2ar^2+\cos(2c+2br^2)+1\}^2}\,[2ar^2+4br^2\sin(2c+2br^2)+3\cos(2c+2br^2)+3]$$

$$8\pi p_r=\frac{2a\{2ar^2+\cos(2c+2br^2)+1\}^{-1}}{aB\{f_1(r)-f_2(r)\}-2\sqrt{a}\,Ab}\,[b\cos(c+br^2)\{2\sqrt{a}\,A\sec(c+br^2)-4B\}+aB\{f_2(r)-f_1(r)\}]$$

$$\Delta=\frac{4ar[a-b\sin(2c+2br^2)]}{[2ar^2+\cos(2c+2br^2)+1]^2}\,\left[aB\{f_1(r)-f_2(r)\}-2\sqrt{a}\,A\,b\right]^{-1}[aBr\{f_1(r)-f_2(r)\}+2b\cos(c+br^2)\{Br-\sqrt{a}\,A\,r\sec(c+br^2)\}]$$

$$8\pi p_t=8\pi p_r+\Delta\,.$$

Here

$$f_1(r)=\ln\left[\cos\left(\frac{c+br^2}{2}\right)-\sin\left(\frac{c+br^2}{2}\right)\right]$$

$$f_2(r)=\ln\left[\cos\left(\frac{c+br^2}{2}\right)+\sin\left(\frac{c+br^2}{2}\right)\right].$$

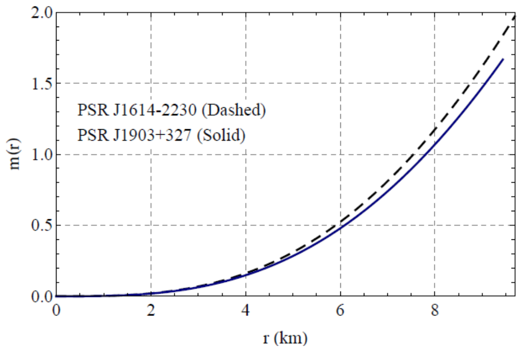

Fig 5. Variation of mass function.

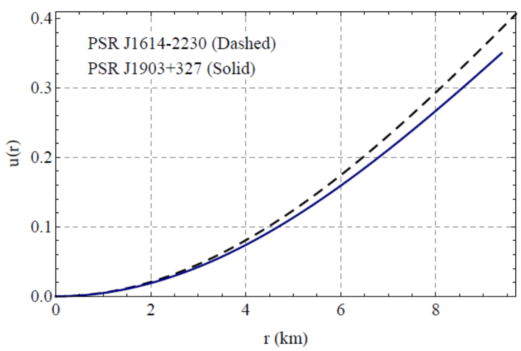

Fig 6. Variation of compactness factor.

IV. PHYSICAL PROPERTIES

The solution has to be free from any singularity for a compact star configuration. The central pressure and density are given as

$$8\pi p_{rc} = 8\pi p_{tc} = \frac{2a\{1+\cos 2c\}^{-1}}{aB\{f_1(0)-f_2(0)\}-2\sqrt{a}\,Ab} \times$$

$$\left[b\cos c\,\left\{2\sqrt{a}\,A\sec c - 4B\right\} + 2B\{f_2(0)-f_1(0)\}\right]$$

$$8\pi\rho = \frac{6a}{1+\cos 2c} > 0 \quad \forall \;\; a > 0$$

This central values are non-zero as long as $a > 0$ and doesn't tends to infinity implying that the solution is non-singular. Further, the solution must satisfy the Zeldovich's criterion i.e. $p_{rc}/\rho_c \leq 1$ for any physical matter. The non-singularity and the Zeldovich's condition leads to

$$\frac{1}{2\sqrt{a}\,b}[4b\cos c - af_1(0) + af_2(0)] < \frac{A}{B}$$

$$\leq \frac{1}{2\sqrt{a}\,b}[4b\cos c + af_1(0) - af_2(0)]\,.$$

Now the mass function, compactness and the redshift can be found as

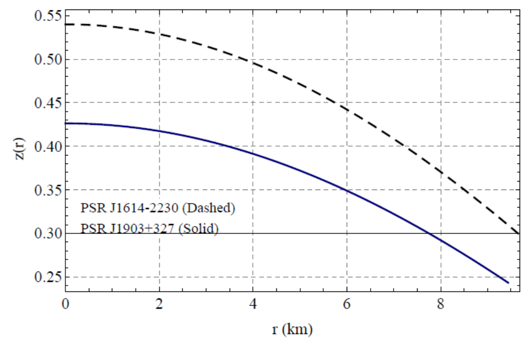

Fig 7. Variation of redshift.

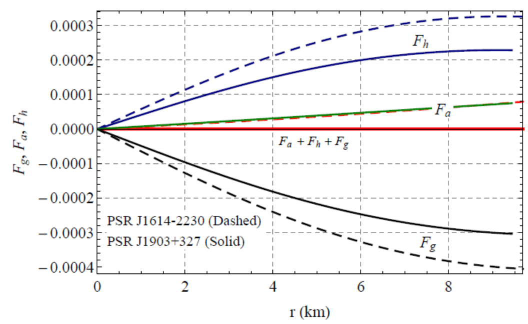

Fig 8. Variation of forces in TOV-equation.

$$m(r) = \frac{ar^3}{1+2ar^2+\cos(c+2br^2)}$$

$$u = \frac{2m(r)}{r} = \frac{2ar^2}{1+2ar^2+\cos(c+2br^2)}$$

$$z(r) = e^{-\nu/2} - 1$$

V. BOUNDARY CONDITIONS

The interior spacetime has to be match smoothly with the exterior Schwarzschild for the continuity of the spacetime fabric. The exterior metric is takes as

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$-r^2(d\theta^2 + \sin^2\theta\;d\phi^2)$$

The two metrics has a boundary at $r = R$ and after matching we get

$$e^{\nu(R)} = 1 - \frac{2M}{R} = e^{-\lambda(R)}\;\;.$$

Further, the boundary of a compact star is set when the pressures is zero. Using these conditions we get

$$A = \frac{\sec(c+bR^2)}{4b}\sqrt{1-\frac{2M}{R}}\ [4b\cos(c+bR^2)$$
$$+af_1(R) - af_2(R)]$$

$$B = \frac{2\sqrt{a}\,Ab}{4b\cos(c+bR^2)+af_1(R)-af_2(R)}$$

$$a = \frac{1}{R^2\sec^2(c+bR^2)}\left[\frac{1}{1-2M/R}-1\right].$$

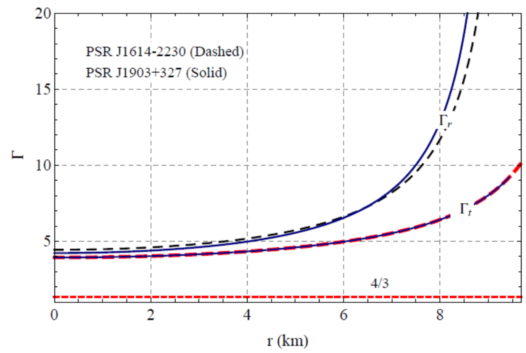

Fig 9. Variation of adiabatic index.

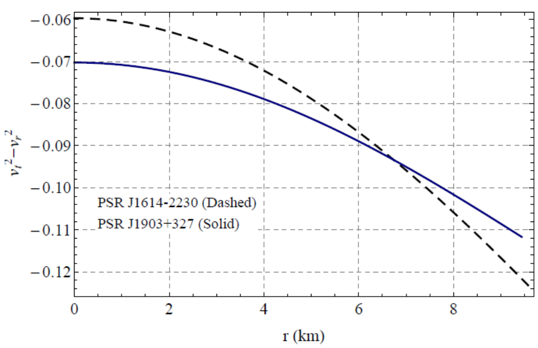

Fig 10. Variation of stability factor

VI. STABILITY AND EQUILIBRIUM CONDITIONS

The solution to represent a configuration in equilibrium, one must satisfy the TOV-equation given by

$$-\frac{\nu'}{2}(\rho+p_r) - \frac{dp_r}{dr} + \frac{2\Delta}{r} = 0\ .$$

The first term is gravity, second the hydrostatic and the last term is anisotropic force. Further, the stability can also be seen via two physical parameters, adiabatic index [10]

$$\Gamma = \frac{\rho+p_r}{p_r}\frac{dp_r}{d\rho}\ .$$

The stability factor $v_t^2 - v_r^2$, where $v_r^2 = dp_r/d\rho$ and $v_t^2 = dp_t/d\rho$, the speed sounds. For stability from gravitation collapse, $\Gamma > 4/3$ for positive anisotropy [11] and the stability factor must lie within -1 and 0 [12]. Further, the stellar system must also be stable under radial perturbations. This happens only if the mass is an increasing function its central density or $\frac{dM}{d\rho_c} > 0$ [13,14].

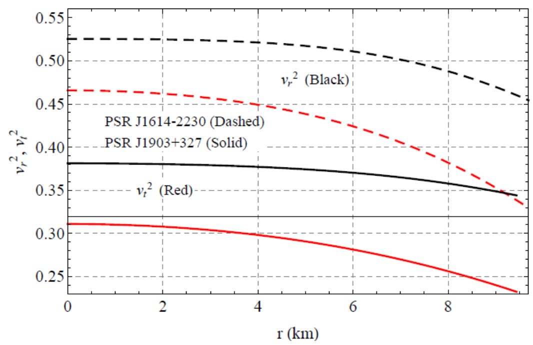

Fig 11. Variation of speed of sound.

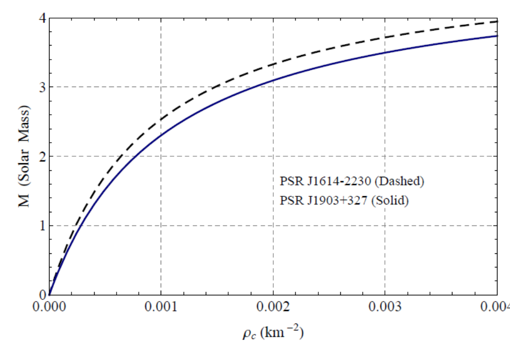

Fig 12. Variation of mass in central density.

The expression for mass is given by

$$M = \frac{4\pi\rho_c R^3(1+\cos 2c)/3}{1+\frac{8\pi}{3}\rho_c R^2(1+\cos 2c)+\cos(2c+2bR^2)}\ .$$

VII. CONCLUSION

We have presented a model of compact star via an exact solution method. This solution was tested through various physical conditions and found to satisfy all the physically acceptability criteria. The matching of the interior metric with the exterior vacuum can be seen in Fig. 1. The variation of density and pressures are shown in Figs. 2 and 3. From these figures it is clear that the central values are not infinite i.e. non-singular. The pressure anisotropy variation in provided in Fig. 4. The central region of the stellar system has vanishing anisotropy i.e. $p_r(0) = p_t(0)$. The mass function trend is given in Fig. 5 showing an increasing function of the radial coordinate. The compactness factor is given in Fig. 6 where one can see the surface value is less than 8/9. This means that the solution fulfills the Buchdahl limit hence free from gravitational collapse. The surface redshift can be seen from Fig. 7. The equilibrium of the system can be seen in Fig. 8 where all the force components are balanced at the interior. Further, the central value of the adiabatic index is clearly greater than 4/3 (Fig. 9), implying that the system is stable under perturbations. This claim is further strengthen by fulfilling the Abreu et al. criterion where the stability lies within -1 and 0, Fig. 10. The satisfaction of causality is a must for all physically acceptable systems. This model does satisfy the causality condition as the speed of sound is sub-luminal (Fig. 11). Lastly, the solution must also test its stability under density perturbation. The static stability criterion requires the mass as an increasing function of its central density. Figure 12 shows the satisfaction of static stability criterion. All the graphs were generated for two compact stars, PSR J1614-2230 ($a = 0.002533/km^2, 0.0015/km^2, c = 0.8$) and PSR J1903+327 ($a = 0.0018513/km^2, 0.001/km^2, c = 0.9$). Since this solution is compatible with observed compact stars, it might have astrophysical applications in future.

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