

Mathematical study of Concentration Distribution in Miscible fluid flow through Homogeneous Porous Media

Dr. K. Shivashankara¹, Venu Prasad K.K.², Prakasha P.³, Dhananjaiiah D.S.⁴

¹Associate Professor, Head Department of Mathematics, Yuvraja's college university of Mysore, Mysore

^{2,4}Assistant Professor, Department of Mathematics, Government First Grade college, K.R.Pet Mandya.
India

³Assistant Professor, Department of Mathematics, Government College for women, Maddur Mandya.
India

Abstract - The governing equation of longitudinal dispersion phenomenon of one-dimensional concentration distribution in miscible fluid flow through homogeneous porous media has been studied in term of advection-dispersion equation. This one-dimensional equation has been solved analytically by using Duhamel's theorem, Laplace transform and moving coordinates with appropriate initial and boundary condition. The final solution is obtained in terms of exponential form as well as complementary error function and it is concluded that the concentration distribution of miscible fluids decreases with time. The developed analytical solutions may support as a valuable tool for evaluating the aquifer concentration at any position and time.

Index Terms - Advection, dispersion, Integral transforms, Moving coordinates, Duhamel's theorem.

INTRODUCTION

The modern civilization producing large amount of wastes often create a hazard to the groundwater quality and before now have resultant in many incidents of groundwater pollution. Degradation of groundwater quality can occur in huge areas by miscible fluid flow through porous media from plane or diffuse sources like deep percolation from intensively farmed fields, or it can be caused by point sources such as mine spoils, septic tank, cemeteries, garbage disposal sites, and oil spoils or other unintentional entry of pollutants into the underground environment. Any more possibility is pollution by like seepage from polluted streams or intrusion of salt

water, line sources of poor quality water from oceans in coastal regions.

The first fluid is displaced in a porous medium by a new fluid with the first fluid this is known as miscible displacement process. The phenomenon of miscible displacement can be seen in coastal areas, where the fresh water is step by step displaced by contaminant or salt water and a transition zone develops between fresh water and salt water. This phenomenon in porous media plays an outstanding role in many science and engineering fields like the process of oil recovery in petroleum engineering, pollution of groundwater by waste product disposed underground movement of mineral in the soil and recovery of spent liquors in pulping process and many problems which involves flow in porous media. The several researchers studied the analytical solution for one-dimensional advection dispersion equation in semi-infinite or finite porous media, mainly, Ogata and Banks [3] for a constant input concentration. Eneman et al. [15] gives the analysis for the systems where fresh water is overlain by water with a higher density in coastal delta areas. Meher [16] and Mehta [17] considered the Dispersion of Miscible fluid in semi infinite porous media with unsteady velocity distribution using a domain decomposing method An exact solution of the linear advection-dispersion transport equation with constant coefficients was introduced by Perez et al. [24] for both transient and steady state regimes and classic version of Generalized Integral Transform Technique was used in solving analytically. S R Sudheendra

[26-28] Discussed the Mathematical Solutions of transport of pollutants through unsaturated porous media with adsorption in a finite domain and he gives the Mathematical Analysis of transport of pollutants through unsaturated porous media with adsorption and radioactive decay also gives the Mathematical Analysis of Solute Transport Exponentially Varies With Time In Unsaturated porous

Mathematical Model

The one dimensional Advection-Dispersion equation describes the concentration distribution of miscible fluids that means salt water or contaminated with fresh water, flow in homogeneous porous media along with initial and boundary conditions can be written as

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial z^2} - w \frac{\partial C}{\partial z} - \left(\frac{1-n}{n} \right) \frac{\partial S}{\partial t}$$

The equilibrium isotherm between solution and adsorbed phase is given by $\frac{\partial S}{\partial t} = K_d \frac{\partial C}{\partial t}$, K_d is the distribution coefficient

$$\begin{aligned} \frac{\partial C}{\partial t} &= D \frac{\partial^2 C}{\partial z^2} - w \frac{\partial C}{\partial z} - \left(\left(\frac{1-n}{n} \right) K_d \right) \frac{\partial C}{\partial t} \\ \left(1 + \left(\frac{1-n}{n} \right) K_d \right) \frac{\partial C}{\partial t} &= D \frac{\partial^2 C}{\partial z^2} - w \frac{\partial C}{\partial z} \end{aligned}$$

$$\begin{aligned} R \frac{\partial C}{\partial t} &= D \frac{\partial^2 C}{\partial z^2} - w \frac{\partial C}{\partial z} \end{aligned}$$

Let us take $D_1 = \frac{D}{R}$ and $W_1 = \frac{w}{R}$ where $R = \left(1 + \left(\frac{1-n}{n} \right) K_d \right)$, initially, $C=0$, because saturated

flow of fluid of concentration takes place in the porous media.

$$\begin{aligned} \frac{\partial C}{\partial t} &= D_1 \frac{\partial^2 C}{\partial z^2} - w_1 \frac{\partial C}{\partial z} \end{aligned} \tag{1}$$

Thus, the suitable boundary conditions for the given mathematical model

$$\begin{cases} C(z, 0) = 0 & z \geq 0 \\ C(0, t) = C_0 e^{-\gamma t} & t \geq 0 \\ C(\infty, t) = 0 & t \geq 0 \end{cases} \tag{2}$$

The problem then is to describe the concentration as a function of t and z. Where the input condition is assumed at the origin and a flux type homogeneous condition is assumed. Initial concentration is C_0 . To reduce equation (3) to a more familiar form, we considered

$$C(z, t) = \Gamma(z, t) \text{Exp} \left[\frac{w_1 z}{2D_1} - \frac{w_1^2 t}{4D_1} \right] \tag{3}$$

Substitution of (3) reduces equation (1) to Fick's law of diffusion equation

$$\frac{\partial \Gamma}{\partial t} = D_1 \frac{\partial^2 \Gamma}{\partial z^2} \tag{4}$$

Equation (4) is transformed by (1) and its initial and boundary conditions (5) are transformed by (2), the above transformations occur when we substitute equation (3) to equation (1) and its boundary conditions (2) respectively.

$$\begin{cases} \Gamma(0, t) = C_0 \text{Exp} \left[\frac{w_1^2 t}{4D_1} - \gamma t \right] & t \geq 0 \\ \Gamma(z, 0) = 0 & z \geq 0 \\ \Gamma(\infty, t) = 0 & t \geq 0 \end{cases} \tag{5}$$

Differential Equation (4) may be solved. When a time dependent influx of the fluid at $z = 0$. The solution of equation (4) may be obtained readily by use of Duhamel's theorem. [Carslaw and Jeager, 1947].

If $C = F(x, y, z, t)$ is the solution of the diffusion equation where media is semi-infinite with taking initial concentration is zero, its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature $\phi(t)$ is

$$C = \int_0^t \phi(\tau) \frac{\partial}{\partial t} F(x, y, z, t - \tau) d\tau$$

This theorem is utilized principally for heat conduction problems, but the above has been

specialized to fit this specific case of interest. Consider now the problem in which initial concentration is zero and the boundary is maintained at concentration unity. The boundary conditions are

$$\left. \begin{aligned} \Gamma(0, t) &= 0 & t \geq 0 \\ \Gamma(z, 0) &= 1 & z \geq 0 \\ \Gamma(\infty, t) &= 0 & t \geq 0 \end{aligned} \right\}$$

When we Applying Laplace transform to equation (4) is transformed to an ordinary differential equation, that is,

$$\mathcal{L}\left[\frac{\partial \Gamma}{\partial t}\right] = \mathcal{L}\left[D_1 \frac{\partial^2 \Gamma}{\partial z^2}\right].$$

$$\frac{\partial^2 \bar{\Gamma}}{\partial z^2} = \frac{p}{D_1} \bar{\Gamma} . \tag{6}$$

Where $\bar{\Gamma}(z, p) = \int_0^\infty e^{-pt} \Gamma(z, t) dt$ and p is the Laplace parameter .

The solution of the above differential equation (6)

is $\bar{\Gamma} = Ae^{-qz} + Be^{qz}$ where, $q = \pm \sqrt{\frac{p}{D_1}}$ when we

apply boundary condition as $z = 0$ we get $A = \frac{1}{p}$ and

for the boundary condition as $z \rightarrow \infty$ we get $B = 0$ thus the particular solution of the Laplace transformed equation is

$$\bar{\Gamma} = \frac{1}{p} e^{-qz}$$

The inversion Laplace transforms of the above function is gives, the result

$$\Gamma = 1 - \operatorname{erf}\left(\frac{z}{2\sqrt{D_1 t}}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{D_1 t}}}^{\infty} e^{-\eta^2} d\eta.$$

Applying Duhamel's theorem, we get solution with the time dependent surface condition at $z = 0$ and initial concentration zero is

$$\Gamma = \int_0^t \phi(\tau) \frac{\partial}{\partial t} \left[\frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{D_1(t-\tau)}}}^{\infty} e^{-\eta^2} d\eta \right] dt.$$

We know that $e^{-\eta^2}$ is a continuous function and we may differentiate under the integral, that is

$$\frac{2}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_{\frac{z}{2\sqrt{D_1(t-\tau)}}}^{\infty} e^{-\eta^2} d\eta = \frac{z}{2\sqrt{D_1 t(t-\tau)^{3/2}}} \operatorname{Exp}\left[\frac{-z^2}{4D_1(t-\tau)}\right].$$

The solution is

$$\Gamma = \frac{z}{2\sqrt{\pi D_1}} \int_0^t \phi(\tau) \operatorname{Exp}\left[\frac{-z^2}{4D_1(t-\tau)}\right] \frac{d\tau}{(t-\tau)^{3/2}}. \tag{7}$$

Putting $\mu = \frac{z}{2\sqrt{D_1(t-\tau)}}$ then the equation (7) can be written as

$$\Gamma = \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{D_1 t}}}^{\infty} \phi\left(t - \frac{z^2}{4D_1 \mu^2}\right) e^{-\mu^2} d\mu. \tag{8}$$

Since $\phi(t) = C_0 \operatorname{Exp}\left(\frac{w_1^2 t}{4D_1} - \gamma t\right)$, The particular solution may be written as

$$\Gamma(z, t) = \frac{2C_0}{\sqrt{\pi}} \operatorname{Exp}\left(\frac{w_1^2 t}{4D_1} - \gamma t\right) \left\{ \int_0^\infty \operatorname{Exp}\left(-\mu^2 - \frac{\epsilon^2}{\mu^2}\right) d\mu - \int_0^\alpha \operatorname{Exp}\left(-\mu^2 - \frac{\epsilon^2}{\mu^2}\right) d\mu \right\}. \tag{9}$$

Where, $\alpha = \frac{z}{2\sqrt{D_1 t}}$ and $\epsilon = \sqrt{\frac{w_1^2}{4D_1} - \gamma} \left(\frac{z}{2\sqrt{D_1}}\right)$

Integrating integral part of the equation (9) the solution is

$$\Gamma(z, t) = \frac{2C_0}{\sqrt{\pi}} \operatorname{Exp}\left(\frac{w_1^2 t}{4D_1} - \gamma t\right) \cdot \left[e^{2\epsilon} \operatorname{erfc}\left(\alpha + \frac{\epsilon}{\alpha}\right) + e^{-2\epsilon} \operatorname{erfc}\left(\alpha - \frac{\epsilon}{\alpha}\right) \right]. \tag{10}$$

From equations (3) and (10)

$$\frac{c}{c_0} = \frac{1}{2} \operatorname{Exp}\left[\frac{w_1 t}{2D_1} - \gamma t\right] \left[e^{2\epsilon} \operatorname{erfc}\left(\alpha + \frac{\epsilon}{\alpha}\right) + e^{-2\epsilon} \operatorname{erfc}\left(\alpha - \frac{\epsilon}{\alpha}\right) \right]. \tag{11}$$

Re-substituting for α and ϵ in equation (11)

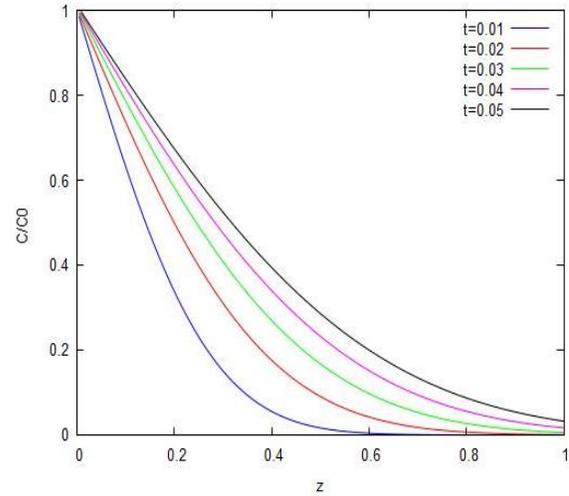
$$\frac{c}{c_0} = \frac{1}{2} \left\{ \operatorname{erfc}\left[\frac{z + t\sqrt{w_1^2 - 4D_1\gamma}}{2\sqrt{D_1 t}}\right] \operatorname{Exp}\left[\frac{w_1 t - 2D_1\gamma t + z\sqrt{2w_1^2 - 4D_1\gamma}}{2D_1}\right] + \operatorname{erfc}\left[\frac{z - t\sqrt{w_1^2 - 4D_1\gamma}}{2\sqrt{D_1 t}}\right] \operatorname{Exp}\left[\frac{w_1 t - 2D_1\gamma t - z\sqrt{2w_1^2 - 4D_1\gamma}}{2D_1}\right] \right\} \tag{12}$$

Here $D_1 = \frac{D}{R}$, $W_1 = \frac{w}{R}$ and $R = \left(1 + \left(\frac{1-n}{n}\right) K_d\right)$.

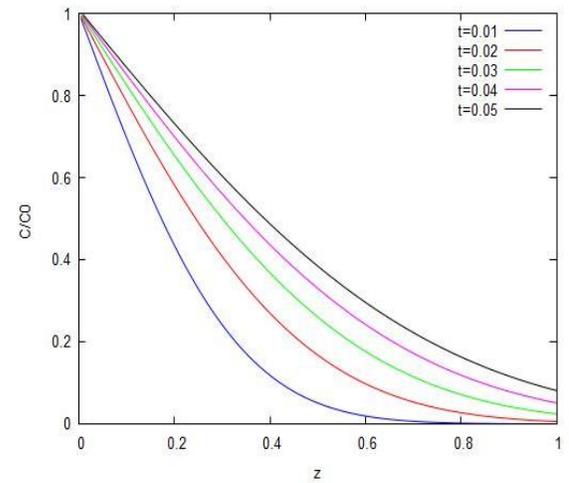
RESULTS AND DISCUSSION

The equation (12) represents concentration profile c/c_0 of the miscible fluid i.e. salt water or contaminated with fresh water, for any time t and distance z , the solution is obtained by using analytical method and We obtain the solution in terms of exponential form as well as complementary error function. The major constraints of the analytical methods are that the geometry of the problem must be regular. The applicability is for comparatively easy problems. This method is bit more fit than other standard methods for one dimensional transport problems. Following graphs represents the Break-through-curve for concentration profiles and distance along the porous media with porosity n . the ratio c/c_0 decreases with depth as n decreases the effect of distributive coefficient K_d , a velocity w and dispersion coefficient D whereas concentration profile versus time and for different values of depth z .

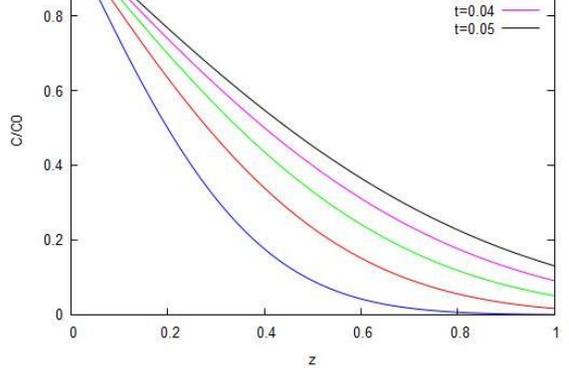
Figures 1 to 4 represents the concentration profiles verses time in the porous media for depth z . It is seen that for a different velocity $w = 0.85\text{m/day}$, $w = 1.1\text{ m/day}$, $w = 1.35\text{ m/day}$, $w = 1.60\text{ m/day}$ with respect to the dispersion coefficient $D = 1.3\text{ m}^2/\text{day}$, $D = 2.18\text{ m}^2/\text{day}$, $D = 3.28\text{ m}^2/\text{day}$, $D = 4.30\text{ m}^2/\text{day}$ and distribution coefficient K_d , C/C_0 decreases with depth as porosity n decreases due to the distributive coefficient K_d and if time increases the concentration decreases for different time.



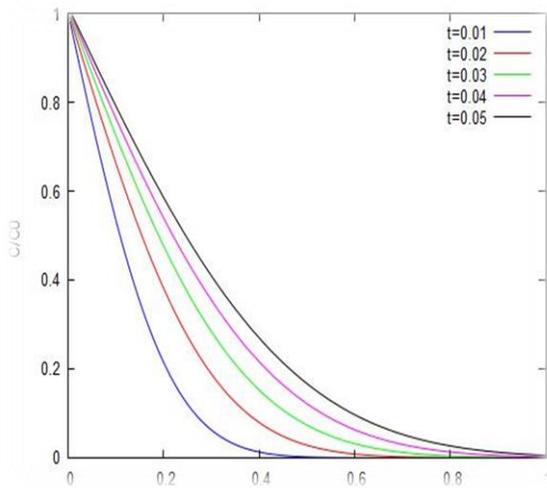
$D=1.3\text{ m}^2/\text{day}$ & $W=0.85\text{ m/day}$ $D=2.18\text{m}^2/\text{day}$ & $W=1.1\text{ m/day}$



$D=3.28\text{m}^2/\text{day}$ & $W=1.35\text{ m/day}$ $D=4.30\text{m}^2/\text{day}$ & $W=1.60\text{ m/day}$



$D=4.30\text{m}^2/\text{day}$ & $W=1.60\text{ m/day}$



CONCLUSION

This mathematical model is dependable with physical phenomenon of the longitudinal dispersion of salt water or contaminated in homogenous porous medium. From the above graphs we can conclude that concentration of the salt water or contaminated is decrease as the distance z and time t increases. Laplace transform technique is a very useful method to obtain analytical solutions for solute transport in homogeneous porous media and under different flow atmosphere. The analytical expressions derived here are handy to the study of salinity intrusion in groundwater, useful in making quantitative predictions on the possible Contamination of groundwater supplies resulting from groundwater movement through buried wastes.

REFERENCE

- [1] Bastian, W.C.; and Lapidus, L. (1956). Longitudinal diffusion in ion exchange and chromatographic columns. *The Journal of Physical Chemistry*, 60(6), 816-817
- [2] Ebach, E.H. & White, R. *Mixing of Fluids Flowing through Beds of Packed Solids*, A. J. Ch. E., 4, p. 161, 1958.
- [3] Ogta, A. & Banks, R.B., A Solution of Differential Equation of Longitudinal Dispersion in Porous Media, *Profess, Paper No. 411-A*, 1961.
- [4] Hoopes, J.A. & Harteman, D.R.F., *Wastewater Recharge and Dispersion in Porous Media*, Technical Report No. 75, 1965.
- [5] Bruce, J.C. & Street, R.L., *Studies of Free Surface Flow and Two-Dimensional Dispersion in Porous Media*, Report No. 63, Civil Eng. Dept. Stanford Uni., Stanford California, 1966.
- [6] Marino, M.A., *Flow Against Dispersion in Non-Adsorbing Porous Media*, *J. Hydrology*, 37, pp. 149-158, 1978.
- [7] Al-Niami, A.N.S. & Rushton, K.R., *Analysis of Flow Against Dispersion in Porous Media*, *J. Hydrology*, 33, pp. 87-97, 1977.
- [8] Basak, P., *Evaporation from Horizontal Soil Columns with Variable Diffusivity*, *J. Hydrology*, 39, pp. 120-136, 1978.
- [9] Hunt, B., *Dispersion Calculations in Non-uniform Seepage*, *J. Hydrology*, 36, pp. 261-277, 1978.
- [10] Wang, St. T., McMillan, A.F. & Chen B.H., *Dispersion of Pollutants in Channels with Non-Uniform Velocity Distribution*, *Water Research*, 12, pp. 389-394, 1978.
- [11] Kumar, N., *Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution*, *Nordic Hydrology*, pp. 167-178, 1983.
- [12] Mehta, M.N. & Patel, T.A., *A Solution of Burger's Equation for Longitudinal Dispersion of Miscible Fluid Flow through Porous Media*, *Indian Journal of Petroleum Geology*, 14(2), pp. 49-54, 2005.
- [13] Mehta, M.N. & Saroj Yadav, *Classical Solution of Non-Linear Equations Arising in Fluid Flow through Homogenous Porous Media*, PhD thesis submitted to SVNIT, Surat, 2009
- [14] Rudraiah N. and Chiu-On Ng., *Dispersion in Porous Media with and without Reaction: A Review*. *J. of Porous Media*, 3, 2007.
- [15] Eeman S., Van Der Zee, S. E. A. T. M., Leijnse A., De Louw, P. G. B and Maas C., *Response to recharge variation of thin lenses and their mixing zone with underlying saline groundwater* *Hydrol. Earth Syst. Sci. Discuss.*, 16, 2012, 355-349.
- [16] Meher R. K. Mehta M. N. Meher S. K., *Adomian Decomposition Method for Dispersion Phenomena Arising in Longitudinal Dispersion of Miscible Fluid Flow through Porous Media*, *Adv. Theor. Appl. Mech.*, 3(5), 2010, 211-220.
- [17] Mehta M. N., *A singular perturbation solution of one-dimensional flow in unsaturated porous media with small diffusivity coefficient*, *Proc. FMFP E1 to E4*, 1975.
- [18] Moreira D. M., Vilhena M. T., Buske D., and Tirabassi T., *The state-of-art of the ILTT method to simulate pollutant dispersion in the atmosphere*, *Atmos. Res.*, 92, 2009, 1-17.
- [19] Moreira D. M., Vilhena M. T., Buske D., and Tirabassi T., *The GILTT solution of the*

- advection-diffusion equation for an in homogenous and non-stationary PBL, Atmos. Environ., 40, 2006, 3186-3194.
- [20] Jaiswal, D.K., Kumar A., Kumar N. and Singh M. K., Solute Transport along Temporally and Spatially Dependent flows through Horizontal Semi-infinite Media: Dispersion being proportional to square of velocity, Journal of Hydrologic Engineering (ASCE). 16(3) 2011, 228-238.
- [21] Jaiswal, D.K., Kumar A., Kumar N., Yadav R. R., Analytical solutions for temporally and spatially dependent solute dispersion of pulse type input concentration in one-dimensional semi-infinite media, Journal of Hydro-environmental Research 2, 2009, 254-263.
- [22] Yadav R. R., Jaiswal D. K., Yadav H. K., and Gulrana, One-dimensional temporally dependent advection-dispersion equation in porous media, Analytical solution, Natural Resource Modelling., 23(12), 2010, 521-539.
- [23] Dong-mei, Sun, Yue-ming, Zhu and Ming-jin, Zhang, Unsteady seepage problems due to water level fluctuation, Journal of Tian Jin University, Vol. 40(7), 2007, 779-785.
- [24] Perez Guerrero J. S., Pimentel L. C. G., Skaggs T. H. and Van Genuchten M. th., Analytical solution of the advection-diffusion transport equation using a change-of-variable and integral transform technique, International Journal of heat and Mass Transfer, 52, 2009, 3297-3304.
- [25] Chen J. S., and Liu C. W. Generalized analytical solution for advection-dispersion equation in finite spatial domain with arbitrary time-dependent inlet boundary -condition, Hydro. Earth Syst. Sci., 8, 2011, 4099-4120.
- [26] Sudheendra S.R., Raji J, & Niranjana CM, Mathematical Solutions of transport of pollutants through unsaturated porous media with adsorption in a finite domain, Int. J. of Combined Research & Development, Vol. 2, No. 2, 32-40. (2014)
- [27] Sudheendra S.R., Praveen Kumar M. & Ramesh T. Mathematical Analysis of transport of pollutants through unsaturated porous media with adsorption and radioactive decay, Int. J. of Combined Research & Development, Vol. 2, No. 4, 01-08.(2014)
- [28] Sudheendra S.R., Raji J, Ramesh T & Venu Prasad K K, 2015 Mathematical Analysis of Solute Transport Exponentially Varies with Time In Unsaturated porous Media, Int. Journal of Engineering Research & Technology, vol.3, No.19,281-285