Commutativity of Near Rings

Dr.Premila Ambaraya
Assistant Professor of Mathematics, AMTA Govt First Grade College Aland, Karnataka India

***Abstract* - A Recent work on near rings has been concerned with sufficient conditions for near rings to be commutative. H.E. Bell proved that if a distributively generated near ring R has an identity**

I. For each x, y in R there exists n(x, y) > 1 such that $(xy-yx)^{m(x,y)} = xy-yx,$ **then R is a commutative ring.**

II. For each x, y in a near ring R there exists + velocity integers $t=t(x,y)\geq 1$ **and** $s=s(x,y)\geq 1$

III. For each x, y in a near ring R there exists + ve integers $t=t(x,y)\geq 1$ **and** $s=s(x,y)\geq 1$ **Such that** $xy=\pm x^t y^s$

In this paper we drop the requirement that R has an identity and show that the other condition is sufficient for R to be commutative. One of the purpose of this paper is to examine the consequence of dropping the hypothesis that R has an identity. Hence we have the following necessary and sufficient condition for a distributively generated near ring to be commutative.

A d.g. near ring R is commutative iff for each x, y in R there exists n > 1 depending on x and y, such that $(xy-yx)^n = xy-yx.$ **also it was shown that a commutative near ring with 1 is a ring. We see that BELL's result follows from our theorem and well known result in ring theory. Heristein proved as a generalization of the Wedderburn theorem that a finite ring is commutative iff each devisor of zero is central. Employing this result we shall also show that a finite d.g. near ring is commutative iff each zero devisor is central.**

***Index Terms* - Near rings d.g. near rings, prime ideals zero symmetric, zero commutative, commutativity of near rings.**

INTRODUCTION

All near rings in this paper are left near rings. The multiplicative center of a near ring R will be denoted by Z (R), the set of nilpotent elements of R denoted by A and the set of idempotent elements of R is denoted by B.

An element x of a near ring R is called distributive if $(a+b)x = ax+bx$ and anti-distributive if $(a+b)x = bx+ax$ for all $a,b\in R$ if all the elements of a near ring R are distributive then R is said to be distributive near ring.

A near ring R is called distributively generated if it contains a multiplicative sub semi group of distributive elements which generates additive group R^+.

A near ring R will be called strongly distributively generated if it contains a set of distributive elements whose square generates R^+.

A near ring R is called zero commutative if $x*y=0 \Rightarrow \ y*x=0$ for $x,y\in R.$

An ideal of a near ring R is defined to be a normal sub group 1 of R^+ such that

1. Rlis a subset of land
2. $(x+i)y-xy\in 1$ for all $x,y\in R$ and $1\in 1$ if R is a d.g. near ring then ii may be replaced by IR is a subset of ?

A near ring ideal P will be called completely prime if $ab\in P \Rightarrow \ a\in P$ or $b\in P$ an element a of the near ring R will be called central if $xa = ax$ for all $x\in R.$

A near ring R is called an N-system if

(i) $xz = yz$ and $z\neq 0 \Rightarrow \ x=y$

(ii) There exists $e\neq 0$ in R such that $e^2 = e$ and

(iii) There exists h in R such that $h+h=e.$

Theorem (Neumann) : The additive group of an N-system is abelian

N-systems are generalizations of near fields for an example of an N-system that is not a near field.

Lemma : Let R be a near ring with no nonzero nilpotent elements. Then Rcontains a family of completely prime ideals with trivial intersection.

Lemma : 2 Let r be a d.g. near ring such that for each x, y in R there exists.

a > 1 depending on x and y such that $(xy-yx)^n = xy-yx.$ Then the set of nilpotent elements is an ideal of R.

Lemma : 3 Let R be a d.g. near ring such that $(R,+)$ is abelian, then R is a ring.

Lemma 4 : Let R be a d.g. near ring with no nonzero devisors of zero and for each x, y in R let there be n > 1.

Depending on x and y such that $(xy-yx)^n = xy-yx$ Then R is commutative.

Proof: If there exists x and y such that $xy-yx \neq 0$ then $(xy-yx)^{n-1} = e$ is a nonzero.

Idempotent. Since R has no nonzero devisors of zero and is d.g., we see that if $w \neq 0$ is a right distributive element of R and r is arbitrary.

Then $e(er-r)=0 \Rightarrow \; er=r$ and $(re-r)w=0 \Rightarrow \; re=r.$ Hence e is an identity of R. In fact any non zero idempotent is e.

Now we wish to show that $(R,+)$ is abelian by demonstrating that either every element of (R, +) is of order two or R is an N-system. Let x, y and z be in R such that xz = yz an $z \neq 0.$ If z is central then $zx = zy \Rightarrow \; x = y.$ If z is not central then there exists $t \neq 0$ such that $zt - tz \neq 0.$ Thus $\{z(zt)-(zt)z\}^{m-1} = e$

Consequently z has right inverse z′. Hence $xz = yz \Rightarrow \; x = xzz' = yzz' = y$

Suppose $e^4 e \neq 0$ and is not central. From above $e+e$ has a right inverse h. since R has no non zero devisors of zero h is also a left inverse. Thus $h(e+e)=e \Rightarrow \; he+he=e.$ Hence R is an N-system and (R, +) is abelian by above theorem.

Suppose $e+e \neq 0$ and is central then $e+e$ is also right distributive. Let $a.b$ be in R. Expanding $(a+b)(e+e)$ by using both distributive laws, we get $a+b=b+a.$ Again $(R,+)$ is abelian If $e+e=0$ then each element of $(R,+)$ is of order two and hence $(R,+)$ is abelian.

Now by lemma 3, R is a ring and the proof is complete.

Theorem: A d.g. near ring R is commutative iff for each $x, y \in R$ there exists $n > 1$ depending on x and y such that $(xy-yx)^n = xy-yx.$

Proof: If every element of R is nilpotent, then clearly R is commutative. IF R has no nonzero nilpotent elements then by lemmal there exists a family of completely prime ideals P with trivial intersection. For each P, R/P is d.g. and has no zero devisors and $(xy-yx)^n = xy-yx$ for x, y in R/P. Thus by lemma 4, R/P is commutative. Hence for each a . b in R $(ab-ba)$ is in P.

Consequently $ab = ba.$

Now let N be the set of nilpotent elements. By lemma 2 N is an ideal of R. now R/N has no nonzero nilpotent elements by the above argument R/N is commutative and hence $(ab-ba)$ is in N for each a.b in R. Thus $ab = ba.$ This completes the proof.

Lemma : Every nontrivial homomarphic image of R contains a non zero central idempotent. Then (R, +) is abelian.

Lemma : Let R be a near ring having zero commutative then annihilator of any non empty subset R is an ideal.

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