Study on the Effect of Urban Pollution Induces by Urban Development Among Forest Territory, Wildlife and Human Population: A Mathematical Model

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Abstract— **Forest are important natural resources. They are continuously decreasing due to overgrowth of human population and urban development. Here, we proposed and analyzed a non-linear mathematical model to study the effects of urban pollution induced by malpractices of urban development in urban forest territory, wildlife population and human population. It is anticipated that pollution concentrations in the environment increase at a constant pace, and that unplanned urban development have a significant impact on forest territory and wildlife species. The resultant model is subjectively examined using differential equation stability theory, as well as quantitatively examined using numerical simulation. It is also being revealed that the concentration of pollutants because of malpractices in urban development would also pose a serious threat resulting into over-exploitation of forest reserves, and wildlife population.**

Index Terms: **forest territory, wildlife population, urban development, urban pollution, stability, numerical analysis.**

1.INTRODUCTION

Rajashekariah (2011) states that urbanization is general demographic process by which cities are expanding. Along with the urban development is an integral part (or segment) of any nation's economic development in the current scenario. However, the speed and scale of urbanization enforces different challenges, like housing, transport systems, basic services and infrastructure and as well as job. Around ninety percent of urban expansion in developing countries is near to vulnerable places cause of informal and unplanned settlements. The expansion of urban land consumption excels population size by as much as fifty percent, which is expected to add 1.2 million Km^2 of new urban built up area to the world in the three decades (By UN, reports). An unorganized city has also raised the high risk of different epidemic or pandemic outbreak problem like Ebola in West Africa and presently Covid-19 in all over the world.

Rapid economic growth comprises especially in developing countries, cities are getting more unorganized, complex and congested. Along with urban development, city's green forest cover is reducing substantially and adversely affects the eco-system. Thus reckless urban activities in the developing countries have also threatened wildlife as well as human population either directly or indirectly. With high scale (increase) of urban development wildlife habitat is also affected badly, for that cause human-wildlife conflict is common incidents in urban fringes (Soulsbury & White, 2015). Human - wildlife clashes are severe issues in most of the parts in India (Vijayraghvan, 2013). Continuous decrease of urban green cover area also aggravates the concentration of different kinds of harmful or fatal pollutants in cities and makes the problem of environmental pollution more critical.

Keeping all important facts, it can be considered that urban development activities are major factors for clearing forest cover area and environmental pollution in cities. So, in demand to achieve sustainability, it is significant to save the urban structure from lethal things due to development actions. It is necessary to make a balance among forest resources, wildlife species, unprecedented growing human population, population pressure induced urban activities or expansion of urban areas and urban pollution.

Quietly mathematical modelling as capable implement for learning the collaborative (mutual) dynamics of inhabitants with green structures. In this field many authors have studied the lethal effect of such accomplishments like industrial development, urban activities and mining etc. has been seen on forest resources and population of biological (like, wildlife and human population) byDubey and Das, 1999; Dubey, Upahayay and Hussain, 2003; Shukla *et al*. (2009); Agarwal, Fatima and Freedman, 2010; Shukla, Lata and Misra 2011; Jyotsna and Tandon, 2017. Some researchers have also introduced population pressure in mathematical model to study the impact of overgrowing inhabitants on forest and living system (Dubey *et al*. 2009; Agarwal and Devi, 2012; Misra, Lata and Shukla, 2014; Tandon, Jyotsna and Dey, 2018). Jyotsna and Tandon (2018) have studied a model to investigate the sustainability of an urban structure in the manifestation of unorganized urban change and disproportionate pollution. They have found that in long run unorganized urban development practices would only responsible for extinction of urban forest and urban population. The effect of excessive rising population and reckless mining activities on forest assets in an urban province have studied by Tandon *et al.* (2017). They have also recommended for sustainable mining; some regulating measure should be enforced by the government. Degradation of environment cause of unprecedented urbanization is also incorporated in a mathematical model by Tandon and Jyotsna (2016). They found that excessive growth in urban-induced pollution would also decline the human population. Jyotsna and Tandon (2017) have investigated a model on the persistence of forest reserveneedy wildlife population, they have found that overutilization of forest assets by human population may lead to extinction of forest assets and wildlife population. So, control measure is prerequisite to mitigate the problem of unorganized urban practices, industrialization, exploitation of forest resources etc. in urban fringe.

However, the sustainability of an urban structure in the manifestation of urban forest territory, growing human and wildlife population with reference to urban development practices induces excessive pollution has not been studied (considered) in the previous non-linear modelling. In this model, it is proposed that the effect of urban pollution induces by malpractices of urban development in forest territory, wildlife population in urban fringe and human population. Here, expansion of urban region is also enforcing by growing human population and population pressure in this model.

2. MATHEMATICAL MODEL

The following proposed mathematical model as system of non-linear differential equation:

$$\frac{dF}{dt} = rF\left(1-\frac{F}{L}\right) - \beta_1 WF - \beta_2 FN - \gamma F^2 U - \alpha_1 FT \quad (2.1a)$$

$$\frac{dW}{dt} = \lambda F + \pi_1 \beta_1 FW - \delta_1 WT - \delta_0 W \quad (2.1b)$$

$$\frac{dN}{dt} = sN\left(1-\frac{N}{K}\right) + \pi_2 \beta_2 FN - \alpha_0 NT \quad (2.1c)$$

$$\frac{dP}{dt} = \eta N - \lambda_0 P \quad (2.1d)$$

$$\frac{dU}{dt} = Q + \kappa_1 P - \theta_0 U \quad (2.1e)$$

$$\frac{dT}{dt} = Q_0 + \theta_1 U - \alpha_2 FT - \mu_0 T \quad (2.1f)$$

Where, initially all variables F, W, N, P, U and T are non-negative with $0 < \pi_1 \leq 1$ and $0 < \pi_2 \leq 1$.

In the model (2.1), F is urban forest territory, W is wildlife population, N is human inhabitants, P is the population or inhabitants pressure of human, U represents urban area or urban development and T is the level of urban pollution in the surroundings.

The included constraints in the model (2.1) are as demarcated in the following table.

Table 2.1

Parameters	Definition
r	Inherent growth rate offorest territory
L	Carrying size of forest territory
β_1	Rate of exhaustion of forest territory due to wildlife population
β_2	Draining rate coefficient of forest territory due to human inhabitants
γ	Exhaustion rate coefficient of forest territory due to urbanization
α_1	Reduction rate coefficient of pollutants due to forest territory
λ	Extension rate coefficient of wildlife population

π_1	part of β_1 taken as the extension rate coefficient of the wildlife population due to forest territory
δ_1	Reduction rate coefficient of wildlife population due to pollutant or toxicant
δ_0	Natural depletion rate of wildlife population
s	Inherent growth rate coefficient of human inhabitants
K	Carrying size of human inhabitants
π_2	Part of β_2 taken as the extension rate coefficient of human inhabitants due to forest territory
α_0	Reduction the rate coefficient of human inhabitants due to pollutant or toxicant
η	Constant growth rate of human inhabitants
λ_0	Natural depletion rate of population or inhabitants pressure of human
Q	Persistent growth rate of urbanization
κ_1	Extension rate coefficient of urbanization due to population pressure
θ_0	Natural depletion rate of urbanization
Q_0	Persistent growth of urban pollution in the surroundings
θ_1	Growth rate coefficient of pollutant due to forest territory
α_2	Reduction rate of urban pollution by forest territory
μ_0	Natural depletion due to pollutant

With these considerations, model has been proposed that the dynamics of urban forest territory is assumed to be increasing logistically and concurrently affected by many genuine reason. It is well known that wildlife and human inhabitants is either directly or indirectly dependent on the forest for its survival. On the other hand, settlement of growing human inhabitants in urban cities, urban expansion is needed that enforce the urban development activities causes clearing the urban forest territory. For this a quadratic term F^2U with exhaustion rate co-efficient γ is incorporated in the model, the term F^2U states also that the carrying size of urban forest territory (Misra *et al*., 2014) affected by vulnerable urban development practices. The terms FW, FN and FT correspondingly, describe the consequence of growing wildlife (forest dependent), human or population density and urban pollutant in the environment are affecting the inherent growth of forest territory. The depletion rate co-efficient of these linear terms FW, FN and FN is β_1, β_2 and α_1 respectively in the differential equation (2.1a) of this model. The growth of wildlife population is dependent upon the forest and the migration of wildlife population to the forest, both considerations are to be involved or incorporated in the differential equation (2.1b) of the model. However, the wildlife population may also have affected with the interaction of urban pollutant. Again, logistic growth of human inhabitants is considered in the differential equation (2.1c) of the model. For the livelihood, urban human inhabitants is also dependent on the forest. It means that the growth of human population increases, along with forest. This urban inhabitants of human is badly affected by the urban pollutant in the surroundings through different ways.

Since, the uninterrupted growing urban population imposes high requirements and orders, after that population force creates (builds) up on the surroundings. It is rises with growth in population. The dynamics of population force has incorporated by Misra *et al*. (2014). So, keeping this views, the growth of population force is to be considered directly comparative to the urban inhabitants, along with its natural running down in the differential equation (2.1d) of the model.

Urban development is a necessary part of the country's economy and it is unending process. With the high rise (increase) of population and population force in urban areas, demands for multifarious like housing, transportation and industries get also increases. With these concepts, the urban development practices are anticipated to be growing at a persistent rate and augmenting through population force and try to control by some external rules for the safety of environment. Underneath, the changing aspects of urban expansion practices remain considered and shown by equation (2.1e).

Additionally, in urban cities diverse external causes generate pollutants in the surroundings, where urban expansion accomplishments (practices) deepen their concentration level. Since, urban forest territory also work as go down for the pollutants. The equation (2.1f) has considered about the pollutant concentrations.

3. QUALITATIVE ANALYSIS

Here, the model (2.1) explored by qualitatively using stability theory of non-lineardifferential equation.

3.1 Region of attraction

Lemma 3.1. The subsequent set of model system (2.1) createsregion of attraction for all solutions startingin the positive orthant.

To explore the model system (2.1), we require the subsequent lemma (Freedman, 1985)

Lemma 3.1*The set*

$$\Omega=\left\{\begin{matrix}(F,W,N,P,U,T):0\le F\le L,0\le W\le W_m,\\ 0\le N\le N_m,0\le P\le P_m,0\le U\le U_m,0\le T\le T_m\end{matrix}\right\}$$

Where, $W_m=\dfrac{\lambda L}{\delta_0-\pi_1\beta_1 L}$ with $\delta_0-\pi_1\beta_1 L>0$,

$N_m=\dfrac{k}{s}(s+\pi_2\beta_2 L-\alpha_0 T_m)$ with $(s+\pi_2\beta_2 L-\alpha_0 T_m)>0$, $P_m=\dfrac{\eta N_m}{\lambda_0}$,

$U_m=\dfrac{Q+k_1P_m}{\theta_0}$, $T_m=\dfrac{Q_0+\theta_1 U_m}{\mu_0}$

Solution of this lemma is discussed in Appendix A.

3.2 Equilibrium Analysis

As of model (2.1), it couldbe easily observed that there are occurs six equilibrium points:

1. $E_0(0,0,0,0,\overline{U},\overline{T})$,
2. $E_1(\hat{F},0,0,0,\hat{U},\hat{T})$,
3. $E_2(\tilde{F},\tilde{W},0,0,\tilde{U},\tilde{T})$,
4. $E_3(0,0,\breve{N},\breve{P},\breve{U},\breve{T})$,
5. $E_4(\overline{\overline{F}},0,\overline{\overline{N}},\overline{\overline{P}},\overline{\overline{U}},\overline{\overline{T}})$,
6. $E_5(F^*,W^*,N^*,P^*,U^*,T^*)$,

The details of these equilibrium points with their existence have been given below.

Existence of $E_0(0,0,0,0,\overline{U},\overline{T})$: It is obvious, where

$$\overline{U}=\frac{Q}{\theta_0} \qquad (3.1)$$

$$\overline{T}=\frac{1}{\mu_0}\left(\frac{Q_0\theta_0+\theta_1 Q}{\theta}\right) \qquad (3.2)$$

Existence of $E_1(\hat{F},0,0,0,\hat{U},\hat{T})$: Taking $W=N=P=0$ in the model system (2.1), we get the subsequent equations:

$$r\left(1-\frac{F}{L}\right)-\gamma FU-\alpha_1 T=0 \qquad (3.3)$$

$$Q-\theta_0 U=0 \qquad (3.4)$$

$$Q_0+\theta_1 U-\alpha_2 FT-\mu_0 T=0 \qquad (3.5)$$

As of Equation (3.4), we obtain

$$U=\frac{Q}{\theta_0} \qquad (3.6)$$

Equation (3.5) give T in terms of F as

$$T=\frac{Q_0\theta_0+\theta_1 Q}{\theta_0(\alpha_2 F+\mu_0)}=f(F) \quad \text{(say)} \qquad (3.7)$$

Now, using Equation (3.6) and (3.7) in Equation (3.3),

$$X(F)\equiv r\left(1-\frac{F}{L}\right)-\gamma F\frac{Q}{\theta_0}-\alpha_1 f(F)=0 \qquad (3.8)$$

Exploring this attained Equation (3.8), we get the subsequentfacts:

1. At $F=0$

$$X(0)=r-\alpha_1 f(0)=r-\alpha_1\left(\frac{Q_0\theta_0+\theta_1 Q}{\theta_0\mu_0}\right)>0 \qquad (3.9)$$

which is positive, provided

$$r-\alpha_1\left(\frac{Q_0\theta_0+\theta_1 Q}{\theta_0\mu_0}\right)>0$$

2. At $F=L$

$$X(L)=-\gamma L\frac{Q}{\theta_0}-\alpha_1 f(L)<0 \qquad (3.10)$$

which is clearly negative as $f(L)>0.$

3. $$X'(L)=-\left(\frac{r}{L}+\gamma\frac{Q}{\theta_0}-\alpha_1\alpha_2\frac{Q_0\theta_0+\theta_1 Q}{\theta_0(\alpha_2 L+\mu_0)^2}\right)<0$$

Hence, $F=\hat{F}$, is single positive root in the interval $(0, L)$ provided $X'(L)$ is also negative in $(0, L)$.

Existence of $E_2(\tilde{F}, \tilde{W}, 0, 0, \tilde{U}, \tilde{T})$: Taking $N = P = 0$ in the model system (2.1), the subsequent Equations are attained:

$$rF\left(1-\frac{F}{L}\right)-\beta_1 WF-\gamma F^2 U-\alpha_0 FT=0 \quad (3.11)$$

$$\lambda F+\pi_1\beta_1 FW-\delta_1 WT-\delta_0 W=0 \quad (3.12)$$

$$Q-\theta_0 U=0 \quad (3.13)$$

$$Q_0+\theta_1 U-\alpha_2 FT-\mu_0 T=0 \quad (3.14)$$

As of Equation (3.13), we obtain

$$U=\frac{Q}{\theta_0} \quad (3.15)$$

As of Equation (3.14), we obtain

$$T=\frac{Q_0+\theta_1\frac{Q}{\theta_0}}{(\alpha_2 F+\mu_0)}=g(F) \text{ (say)} \quad (3.16)$$

As of Equation (3.12), we have

$$W=\frac{\lambda F}{\delta_0+\delta_1 g(F)-\pi_1\beta_1 F}=h(F) \text{ (say)} \quad (3.17)$$

Using Equation (3.15), (3.16), (3.17) in Equation (3.11), we have

$$R(F)\equiv r\left(1-\frac{F}{L}\right)-\beta_1 h(F)-\gamma F\frac{Q}{\theta_0}-\alpha_1 g(F) \quad (3.18)$$

Exploring this attained Equation (3.17), we find the subsequent facts:

1 At $F=0$

$$R(0)\equiv r-\alpha_1 g(0)=r-\alpha_1\left(\frac{Q_0+\theta_1\frac{Q}{\theta_0}}{\mu_0}\right)>0 \quad (3.19)$$

2 At $F=L$

$$R(L)\equiv-\beta_1 h(L)-\gamma L\frac{Q}{\theta_0}-\alpha_1 g(L)<0 \quad (3.20)$$

Which is clearly negative provided the following condition must be satisfied

$$\delta_0+\delta_1 g(L)-\pi_1\beta_1 L>0 \quad (3.21)$$

Hence, $F=\tilde{F}$ is single positive root in the interval $(0, L)$, provided $R'(L)$ is also negative in $(0, L)$.

Existence of $E_3(0, 0, \breve{N}, \breve{P}, \breve{U}, \breve{T})$: Taking $F = W = 0$ in the model system (2.1), the subsequent equations are obtained:

$$sN\left(1-\frac{N}{k}\right)-\alpha_0 NT=0 \quad (3.22)$$

$$\eta N-\lambda_0 P=0 \quad (3.23)$$

$$Q+k_1 P-\theta_0 U=0 \quad (3.24)$$

$$Q_0+\theta_1 U-\mu_0 T=0 \quad (3.25)$$

As of Equation (3.23), we have

$$P=\frac{\eta}{\lambda_0}N=f(N) \text{ (say)} \quad (3.26)$$

As of Equation (3.24), we have

$$U=\frac{1}{\theta_0}(Q+k_1 P)=\frac{1}{\theta_0}\{Q+k_1 f(N)\}=g(N) \text{ (say)} \quad (3.27)$$

As of Equation (3.25), we have

$$T=\frac{1}{\mu_0}(Q_0+\theta_1 U)=\frac{1}{\mu_0}\{Q_0+\theta_1 g(N)\}=h(N) \text{ (say)} \quad (3.28)$$

Now, as of Equation (3.22) and (3.28), we have

$$Y(N)\equiv s\left(1-\frac{N}{k}\right)-\alpha_0 h(N) \quad (3.29)$$

Exploring this attained Equation (3.29), we have the subsequent facts:

1 At $N=0$

$$Y(0)\equiv s-\alpha_0 h(0)=s-\frac{\alpha_0}{\mu_0}\left(\frac{Q_0\theta_0+\theta_1 Q}{\theta_0}\right)>0 \quad (3.30)$$

Provided $s-\frac{\alpha_0}{\mu_0}\left(\frac{Q_0\theta_0+\theta_1 Q}{\theta_0}\right)>0$

2 At $N=N_m=\frac{k}{s}(s+\pi_2\beta_2 L-\alpha_0 T)$

$$Y(N_m)\equiv s-\frac{s}{k}N_m-\alpha_0 h(N_m)=-\pi_2\beta_2 L<0 \quad (3.31)$$

Which is clearly negative.

Hence $N = \breve{N}$ is single positive root, provided $Y'(N)$ must be negative in $(0, N_m)$.

Existence of $E_4(\overline{\overline{F}}, 0, \overline{\overline{N}}, \overline{\overline{P}}, \overline{\overline{U}}, \overline{\overline{T}})$: Taking $W = 0$ in the model system (2.1), the subsequent equations are obtained:

$$r\left(1-\frac{F}{L}\right)-\beta_2 N-\gamma FU-\alpha_1 T=0 \quad (3.32)$$

$$s\left(1-\frac{N}{k}\right)+\pi_2\beta_2 F-\alpha_0 T=0 \quad (3.33)$$

$$\eta N-\lambda_0 P=0 \quad (3.34)$$

$$Q+k_1 P-\theta_0 U=0 \quad (3.35)$$

$$Q_0+\theta_1 U-\alpha_2 FT-\mu_0 T=0 \quad (3.36)$$

As of Equation (3.34), we have

$$P=\frac{\eta}{\lambda_0}N=f_1(N) \quad \text{(say)} \quad (3.37)$$

As of Equation (3.35), we have

$$U=\frac{1}{\theta_0}(Q+k_1 P)=\frac{1}{\theta_0}\{Q+k_1 f_1(N)\}=g_1(N)$$

(say) (3.38)

As of Equation (3.36), we have

$$T=\frac{Q_0+\theta_1 g_1(N)}{\alpha_2 F+\mu_0}=h_1(F,N) \quad \text{(say)} \quad (3.39)$$

We derive the following two isoclines in F and N by substituting the Equations (3.38) and (3.39) in Equations (3.32) and (3.33) for U and T:

$$R_1(F,N)\equiv r\left(1-\frac{F}{L}\right)-\beta_2 N-\gamma F g_1(N)-\alpha_1 h_1(F,N)=0 \quad (3.40)$$

$$R_2(F,N)\equiv s\left(1-\frac{N}{k}\right)+\pi_2\beta_2 F-\alpha_0 h_1(F,N)=0 \quad (3.41)$$

(A) We can deduce the following from Equation (3.40):

(a) When $N=0$, F is specified by the following:

$$T_1(F)=r\left(1-\frac{F}{L}\right)-\gamma F g_1(0)-\alpha_1 h(F,0)$$

$$T_1(F)=r\left(1-\frac{F}{L}\right)-\gamma F\frac{Q}{\theta_0}-\alpha_1\frac{Q_0+\theta_1\frac{Q}{\theta_0}}{\alpha_2 F+\mu_0} \quad (3.42)$$

Now, Equation (3.42) can be analyzed as follows:

(i) $T_1(0)=r-\alpha_1\dfrac{\left(Q_0+\theta_1\frac{Q}{\theta_0}\right)}{\mu_0}$, is positive provided

$$r-\alpha_1\frac{(Q_0\theta_0+Q\theta_1)}{\mu_0\theta_0}>0$$

(ii) $$T_1(L)=-\gamma L\frac{Q}{\theta_0}-\alpha_1\frac{(Q_0\theta_0+\theta_1 Q)}{\theta_0(\alpha_2 L+\mu_0)}<0$$

(iii) $$T_1'(F)=-\frac{r}{L}-\gamma\frac{Q}{\theta_0}+\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}$$

$$=-\left[\frac{r}{L}+\gamma\frac{Q}{\theta_0}-\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}\right]<0$$

Provided,

$$\left\{\frac{r}{L}+\gamma\frac{Q}{\theta_0}-\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}\right\}>0$$

Hence, $T_1(F)=0$ has a single positive root in 0 <F < L.

(b) $F<0 \, as \, N\to\infty$

(c) $$\left(\frac{dN}{dF}\right)_1<0$$

(B) As of Equation (3.41), we can deduce the following:

(a) When $N=0$, F is specified by the subsequent equation:

$$T_2(F)=\pi_2\beta_2 F-\alpha_0 h(F,0)$$

$$T_2(F)=\pi_2\beta_2 F-\alpha_0\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)} \quad (3.43)$$

Now, the Equation (3.43) can be analyzed as follows:

1 $$T_2(0)=-\alpha_0\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0\mu_0}<0$$

2 $$T_2(L)=\pi_2\beta_2 L-\alpha_0\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 L+\mu_0)}>0$$

Provided,

$$\pi_2\beta_2 L - \alpha_0 \frac{(Q_0\theta_0 + Q\theta_1)}{\theta_0(\alpha_2 L + \mu_0)} > 0$$

3
$$T'(F) = \pi_2\beta_2 + \alpha_0\alpha_2 \frac{(Q_0\theta_0 + Q\theta_1)}{\theta_0(\alpha_2 F + \mu_0)^2} > 0$$

Hence, $T_2(F) = 0$ has single positive root in 0 < F < L.

(b) $N > 0 \, as \, F \to \infty$

(c) $\left(\frac{dN}{dF}\right)_2 > 0$

For the uniqueness of $F = \overline{\overline{F}} \, and \, N = \overline{\overline{N}},$ we must have $\left(\frac{dN}{dF}\right)_1 < 0 \, and \left(\frac{dN}{dF}\right)_2 > 0$ in the region 0 < F < L.

From the Equation (3.40), we must have $\left(\frac{dN}{dF}\right)_1 < 0$

For this as we have,

$$R_1(F,N) \equiv r\left(1-\frac{F}{L}\right) - \beta_2 N - \gamma F g(N) - \alpha_1 h(F,N) = 0$$

, since $R_1(F,N) = 0$.

From this, we have

$$\left(\frac{dN}{dF}\right)_1 = -\frac{\left(\frac{\partial R_1}{\partial F}\right)}{\left(\frac{\partial R_1}{\partial N}\right)} = -\frac{\left(\frac{r}{L} + \gamma g(N) + \alpha_1 \frac{\partial h}{\partial F}\right)}{\left(\beta_2 + \gamma F \frac{\partial g}{\partial N} + \alpha_1 \frac{\partial h}{\partial N}\right)} \quad (3.44)$$

Which should be negative for uniqueness.

As of Equation (3.41), we must have

$$\left(\frac{dN}{dF}\right)_2 > 0$$

Since, we have

$$R_2(F,N) \equiv s\left(1-\frac{N}{K}\right) + \pi_2\beta_2 F - \alpha_0 h(F,N) = 0$$

, since $R_2(F,N) = 0$

From this we have

$$\left(\frac{dN}{dF}\right)_2 = -\frac{\left(\frac{\partial R_2}{\partial F}\right)}{\left(\frac{\partial R_2}{\partial N}\right)} = \frac{\left(\pi_2\beta_2 - \alpha_0 \frac{\partial h}{\partial F}\right)}{\left(\frac{s}{k} + \alpha_0 \frac{\partial h}{\partial F}\right)} \quad (3.45)$$

which should be positive for uniqueness.

For the uniqueness of $F = \overline{\overline{F}}$ and $N = \overline{\overline{N}}$, we must have $\left(\frac{dN}{dF}\right)_1 < 0 \, and \left(\frac{dN}{dF}\right)_2 > 0$

From the Equation (3.44), (If the forest is not affected by pollutant , therefore $\alpha_1 = 0$)

Hence,

$$\left(\frac{dN}{dF}\right)_1 = -\frac{\left(\frac{r}{L} + \gamma g(N)\right)}{\left(\beta_2 + \gamma F \frac{\partial g}{\partial N}\right)} < 0$$

which is clearly negative.

Similarly, from the Equation (3.45), (If the human population is not affected by pollutant, therefore $\alpha_0 = 0$)

Hence,

$$\left(\frac{dN}{dF}\right)_2 = \frac{\pi_2\beta_2}{\left(\frac{s}{k}\right)} > 0$$

which is clearly positive

By applying this single value of $F = \overline{\overline{F}}$ and $N = \overline{\overline{N}}$, we find other single value also.

Existent of $E_5(F^*, W^*, N^*, P^*, U^*, T^*)$: In view of this model system (2.1), the subsequent equations are achieved:

$$r\left(1-\frac{F}{L}\right) - \beta_1 W - \beta_2 N - \gamma F U - \alpha_1 T = 0 \quad (3.46)$$

$$\lambda F + \pi_1\beta_1 F W - \delta_1 W T - \delta_0 W = 0 \quad (3.47)$$

$$s\left(1-\frac{N}{K}\right) + \pi_2\beta_2 F - \alpha_0 T = 0 \quad (3.48)$$

$$\eta N - \lambda_0 P = 0 \quad (3.49)$$

$$Q + k_1 P - \theta_0 U = 0 \quad (3.50)$$

$$Q_0 + \theta_1 U - \alpha_2 F T - \mu_0 T = 0 \quad (3.51)$$

Equation (3.49) and (3.50) in turn gives P and U in forms of N as

$$P=\frac{\eta}{\lambda_0}N=f_2(N)$$ (say) (3.52)

$$U=\frac{1}{\theta_0}\left(Q+k_1 f_2(N)\right)=g_2(N)$$ (say) (3.53)

From the Equation (3.51), we have

$$T=\frac{Q_0+\theta_1 g_2(N)}{\alpha_2 F+\mu_0}=h_2(F,N)$$ (say) (3.54)

From the Equation (3.47), we have

$$W=\frac{\lambda F}{\delta_0+\delta_1 h_2(F,N)-\pi_1\beta_1 F}=j_2(F,N)$$ (say) (3.55)

We derive the subsequent two isoclines F and N by substituting Equations (3.52),(3.53), (3.54) and (3.55) in Equations (3.46) and (3.48) for P, U, T and W

$$Z_1(F,N)\equiv r\left(1-\frac{F}{L}\right)-\beta_1 j_2(F,N)-\beta_2 N-\gamma F g_2(N)-\alpha_1 h_2(F,N)=0$$ (3.56)

$$Z_2(F,N)\equiv s\left(1-\frac{N}{K}\right)+\pi_2\beta_2 F-\alpha_0 h_2(F,N)=0$$ (3.57)

As of Equation (3.55), we can deduce the following:

(a) When $N=0$, F is specified by the subsequent equation.

$$V_1(F)=r\left(1-\frac{F}{L}\right)-\beta_1 j_2(F,0)-\gamma F g_2(0)-\alpha_1 h_2(F,0)$$

$$V_1(F)=r\left(1-\frac{F}{L}\right)-\beta_1\frac{\lambda F}{\delta_0+\delta_1\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)}-\pi_1\beta_1 F}-\gamma F\frac{Q}{\theta_0}-\alpha_1\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)}$$

(3.58)

Now, the Equation (3.58) can be analyzed as follows:

1. $$V_1(0)=r-\alpha_1\frac{(Q_0\theta_0+Q\theta_1)}{\mu_0}>0$$

Provided,

$$r-\alpha_1\frac{(Q_0\theta_0+Q\theta_1)}{\mu_0}>0$$

2. $$V_1(L)=-\beta_1\frac{\lambda L}{\delta_0+\frac{\delta_1(\theta_0 Q_0+Q\theta_1)}{\theta_0(\alpha_2 L+\mu_0)}-\pi_1\beta_1 L}-\gamma L\frac{Q}{\theta_0}-\alpha_1\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 L+\mu_0)}<0$$

3. $$V_1'(F)=-\frac{r}{L}-\beta_1 j_2'(F,0)-\gamma\frac{Q}{\theta_0}+\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}$$

$$V_1'(F)=-\left\{\frac{r}{L}+\beta_1 j_2'(F,0)+\gamma\frac{Q}{\theta_0}-\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}\right\}<0$$

Provided,

$$\left\{\frac{r}{L}+\beta_1 j_2'(F,0)+\gamma\frac{Q}{\theta_0}-\alpha_1\alpha_2\frac{(Q_0\theta_0+Q\theta_1)}{\theta_0(\alpha_2 F+\mu_0)^2}\right\}>0$$

Hence, $V_1(F)=0,$ has single positive root in $0 < F < L$

(b) $F<0\, as\, N\to\infty$

(c) $\left(\frac{dN}{dF}\right)_1<0$

(A) As of Equation (3.57), we can deduce the following:

(a) When $N=0$, F is specified by the subsequent equation:

$$V_2(F)=s+\pi_2\beta_2 F-\alpha_0 h_2(F,0)$$

$$V_2(F)=s+\pi_2\beta_2 F-\alpha_0\frac{(Q_0\theta_0+Q\theta_0)}{\alpha_2 F+\mu_0}$$

This Equation can be analyzed as:

1. $$V_2(0)=s-\alpha_0\frac{(Q_0\theta_0+Q\theta_0)}{\mu_0}>0$$

2. $$V_2(L)=s+\pi_2\beta_2 L-\alpha_0\frac{(Q_0\theta_0+Q\theta_0)}{\alpha_2 L+\mu_0}>0$$

Provided,

$$s+\pi_2\beta_2 L-\alpha_0\frac{(Q_0\theta_0+Q\theta_0)}{\alpha_2 L+\mu_0}>0$$

3. $$V_2'(F) = \pi_2\beta_2 + \alpha_0\alpha_2 \frac{(Q_0\theta_0 + Q\theta_1)}{\theta_0(\alpha_2 F + \mu_0)^2} > 0$$

4. Hence, $V_2(F) = 0$ has a single positive root in $0 < F < L$.

(b) $N > 0 \, as \, F \to \infty$

(c) $\left(\frac{dN}{dF}\right)_2 > 0$

For the uniqueness of $F = F^*$ and $N = N^*$, we must have $\left(\frac{dN}{dF}\right)_1 < 0$ and $\left(\frac{dN}{dF}\right)_2 > 0$ in the region $0 < F < L$.

Now from the Equation (3.56), we have

$$Z_1(F,N) \equiv r\left(1 - \frac{F}{L}\right) - \beta_1 j_2(F,N) - \beta_2 N - \gamma F g_2(N) - \alpha_1 h_2(F,N) = 0$$

From this we have

$$\left(\frac{dN}{dF}\right)_1 = -\frac{\left(\frac{\partial Z_1}{\partial F}\right)}{\left(\frac{\partial Z_1}{\partial N}\right)} = -\frac{\left\{\frac{r}{L} + \beta_1\frac{\partial j_2}{\partial F} + \gamma g_2(N) + \alpha_1\frac{\partial h_2}{\partial F}\right\}}{\left\{\beta_1\frac{\partial j_2}{\partial N} + \beta_2 + \gamma F\frac{\partial g_2}{\partial N} + \alpha_1\frac{\partial h_2}{\partial N}\right\}} \quad (3.59)$$

From the Equation (3.57), we must have $\left(\frac{dN}{dF}\right)_2 > 0$

For this as we have,

$$Z_2(F,N) \equiv s\left(1 - \frac{N}{K}\right) + \pi_2\beta_2 F - \alpha_0 h_2(F,N) = 0$$

From this now we have,

$$\left(\frac{dN}{dF}\right)_2 = -\frac{\left(\frac{\partial Z_2}{\partial F}\right)}{\left(\frac{\partial Z_2}{\partial N}\right)} = \frac{\pi_2\beta_2 - \alpha_0\frac{\partial h_2}{\partial F}}{\frac{s}{K} + \alpha_0\frac{\partial h_2}{\partial N}} \quad (3.60)$$

which should be positive for uniqueness.

For the uniqueness of $F = F^*$ and $N = N^*$, we must have $\left(\frac{dN}{dF}\right)_1 < 0$ and $\left(\frac{dN}{dF}\right)_2 > 0$.

From the Equation (3.59), (if the forest is not affected by the pollutant and as well as wildlife population) therefore, $\alpha_1 = 0$, $\beta_1 = 0$. Now we have

$$\left(\frac{dN}{dF}\right)_1 = -\frac{\left\{\frac{r}{L} + \gamma g_2(N)\right\}}{\left\{\beta_2 + \gamma F\frac{\partial g_2}{\partial N}\right\}} < 0$$

which is clearly negative.

Similarly, from the Equation (3.60), (if the human population is not affected by the pollutant $\alpha_0 = 0$

$$\left(\frac{dN}{dF}\right)_2 = \frac{\pi_2\beta_2}{\frac{s}{K}} > 0$$

which is clearly positive.

4. STABILITY ANALYSIS

The sign of the eigenvalues from the Jacobian matrix can be used to identify the local stability behavior of equilibrium points. The general Jacobian matrix for the model system (2.1) is useful in this regard.

$$J = \begin{bmatrix} a_{11} & -\beta_1 F & -\beta_2 F & 0 & -\gamma F^2 & -\alpha_1 F \\ \lambda + \pi_1\beta_1 W & b_{22} & 0 & 0 & 0 & -\delta_1 W \\ \pi_2\beta_2 N & 0 & c_{33} & 0 & 0 & -\alpha_0 N \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2 T & 0 & 0 & 0 & \theta_1 & -\alpha_2 F - \mu_0 \end{bmatrix}$$

where,

$$a_{11} = r - 2\frac{r}{L}F - \beta_1 W - \beta_2 N - 2\gamma FU - \alpha_1 T$$

$$b_{22} = \pi_1\beta_1 F - \delta_1 T - \delta_0$$

$$c_{33} = s - 2\frac{s}{k}N + \pi_2\beta_2 F - \alpha_0 T$$

Taking J_i be the Jacobian matrix relates to the matrix J at the equilibrium $E_i(i = 0,1,2,3,4,5)$, then the investigation of this Jacobian matrix for local stability behavior as follows:

1 The matrix J_0 at the equilibrium point $E_0(0,0,0,0,\bar{U},\bar{T})$ display as,

$$J_0=\begin{bmatrix} r-\alpha_1\bar{T} & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\delta_1\bar{T}-\delta_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s-\alpha_0\bar{T} & 0 & 0 & 0 \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2\bar{T} & 0 & 0 & 0 & \theta_1 & -\mu_0 \end{bmatrix}$$

Since $s-\alpha_0\overline{T}=s-\alpha_0\dfrac{(Q_0\theta_0+Q\theta_1)}{\mu_0\theta_0}>0$ for the existence of E_3,

As a result, at the equilibrium point E_0, we attain one positive eigenvalue of J_0 and the system present unstable behavior in N direction.

2. The matrix J_1 at the equilibrium point $E_1(\hat{F},0,0,0,\hat{U},\hat{T})$ display as,

$$J_1=\begin{bmatrix} r-2\frac{r}{L}\hat{F}-2\gamma\hat{F}\hat{U}-\alpha_1\hat{T} & -\beta_1\hat{F} & -\beta_2\hat{F} & 0 & -\gamma\hat{F}^2 & -\alpha_1\hat{F} \\ \lambda & \pi_1\beta_1\hat{F}-\delta_1\hat{T}-\delta_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s+\pi_2\beta_2\hat{F}-\alpha_0\hat{T} & 0 & 0 & 0 \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2\hat{T} & 0 & 0 & 0 & \theta_1 & -\alpha_2\hat{F}-\mu_0 \end{bmatrix}$$

Since $s+\pi_2\beta_2\hat{F}-\alpha_0\hat{T}>0$ as the existence of N_m.

As a result, at the equilibrium point E_1, we attain one positive eigenvalue of J_1 and the system present unstable behavior in N direction.

3. The matrix J_2 at the equilibrium point of $E_2(\tilde{F},\tilde{W},0,0,\tilde{U},\tilde{T})$ display as,

$$J_2=\begin{bmatrix} r-2\frac{r}{L}\tilde{F}-\beta_1\tilde{W}-2\gamma\tilde{F}\tilde{U}-\alpha_1\tilde{T} & -\beta_1\tilde{F} & -\beta_2\tilde{F} & 0 & -\gamma\tilde{F}^2 & -\alpha_1\tilde{T} \\ \lambda+\pi_1\beta_1\tilde{W} & \pi_1\beta_1\tilde{F}-\delta_1\tilde{T}-\delta_0 & 0 & 0 & 0 & -\delta_1\tilde{W} \\ 0 & 0 & s+\pi_2\beta_2\tilde{F}-\alpha_0\tilde{T} & 0 & 0 & 0 \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2\tilde{T} & 0 & 0 & 0 & \theta_1 & -\alpha_2\tilde{F}-\mu_0 \end{bmatrix}$$

Since $s+\pi_2\beta_2\hat{F}-\alpha_0\hat{T}>0$ as the existence of N_m.

As a result, at the equilibrium point E_2, we attain one positive eigenvalue of J_2 and the system present unstable behavior in N direction.

4. The matrix J_3 at the equilibrium point of $E_3(0,0,\breve{N},\breve{P},\breve{U},\breve{T})$ display as,

$$J_3=\begin{bmatrix} r-\beta_2\breve{N}-\alpha_1\breve{T} & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\delta_1\breve{T}-\delta_0 & 0 & 0 & 0 & 0 \\ \pi_2\beta_2\breve{N} & 0 & s-2\frac{s}{k}\breve{N}-\alpha_0\breve{T} & 0 & 0 & -\alpha_0\breve{N} \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2\breve{T} & 0 & 0 & 0 & \theta_1 & -\mu_0 \end{bmatrix}$$

The characteristic equation for a Jacobian matrix J_3 is attain as:

$$(\psi+\delta_1\breve{T}+\delta_0)(\psi^5+B_1\psi^4+B_2\psi^3+B_3\psi^2+B_4\psi+B_5)=0$$

Where,

$$B_1=\left\{\left(\beta_2+2\frac{s}{k}\right)\breve{N}+(\alpha_1+\alpha_0)\breve{T}+(\theta_0+\mu_0+\lambda_0-r-s)\right\}$$

$$B_2=(\beta_2\breve{N}+\alpha_1\breve{T}-r)\left(2\frac{s}{k}\breve{N}-s+\alpha_0\breve{T}+\lambda_0+\theta_0+\mu_0\right)$$
$$+\left(2\frac{s}{k}+\alpha_0\breve{T}-s\right)(\lambda_0+\theta_0+\mu_0)+\lambda_0(\theta_0+\mu_0)+\theta_0\mu_0$$

$$B_3=(r-\beta_2\breve{N}-\alpha_1\breve{T})\left(2\frac{s}{k}\breve{N}+\alpha_0\breve{T}-s\right)(\lambda_0+\theta_0+\mu_0)+$$
$$\left(r-\beta_2\breve{N}-\alpha_1\breve{T}+2\frac{s}{k}\breve{N}+\alpha_0\breve{T}-s\right)$$
$$(\lambda_0\theta_0+\lambda_0\mu_0+\theta_0\mu_0)+\lambda_0\theta_0\mu_0$$

$$B_4=(r-\beta_2\breve{N}-\alpha_1\breve{T})\left(2\frac{s}{k}\breve{N}+\alpha_0\breve{T}-s\right)$$
$$(\lambda_0\theta_0+\lambda_0\mu_0+\theta_0\mu_0)+\left(r-\beta_2\breve{N}-\alpha_1\breve{T}+2\frac{s}{k}\breve{N}+\alpha_0\breve{T}-s\right)$$
$$\lambda_0\theta_0\mu_0$$

$$B_5=(r-\beta_2\breve{N}-\alpha_1\breve{T})\left(2\frac{s}{k}\breve{N}+\alpha_0\breve{T}-s\right)\theta_0\mu_0\lambda_0-\theta_1k_1\eta\alpha_0\breve{N}$$

If the following conditions are satisfied by the Routh-Hurwitz criteria, then the model system have certainly negative or with negative real part eigenvalues and the system will be stable at this equilibrium point $E_3\left(0,0,\breve{N},\breve{P},\breve{U},\breve{T}\right)$.

If $H_1 = B_1 > 0$, $H_2 = \begin{vmatrix} B_1 & 0 \\ 1 & B_2 \end{vmatrix} > 0$,

$$H_3 = \begin{vmatrix} B_1 & B_3 & 0 \\ 1 & B_2 & 0 \\ 0 & B_1 & B_3 \end{vmatrix} > 0,$$

$$H_4 = \begin{vmatrix} B_1 & B_3 & 0 & 0 \\ 1 & B_2 & B_4 & 0 \\ 0 & B_1 & B_3 & 0 \\ 0 & 1 & B_2 & B_4 \end{vmatrix} > 0$$

and

$$H_5 = \begin{vmatrix} B_1 & B_3 & B_5 & 0 & 0 \\ 1 & B_2 & B_4 & 0 & 0 \\ 0 & B_1 & B_3 & B_5 & 0 \\ 0 & 1 & B_2 & B_4 & 0 \\ 0 & 0 & B_1 & B_3 & B_5 \end{vmatrix} > 0$$

.

5. The matrix J_4 at the equilibrium point $E_4\left(\overline{\overline{F}},0,\overline{\overline{N}},\overline{\overline{P}},\overline{\overline{U}},\overline{\overline{T}}\right)$ display as,

$$J_4 = \begin{bmatrix} a_{11} & -\beta_1\overline{\overline{F}} & -\beta_2\overline{\overline{F}} & 0 & -\lambda\overline{\overline{F}}^2 & -\alpha_1\overline{\overline{F}} \\ \lambda & \pi_1\beta_1\overline{\overline{F}}-\delta_1\overline{\overline{T}}-\delta_0 & 0 & 0 & 0 & 0 \\ \pi_2\beta_2\overline{\overline{N}} & 0 & c_{33} & 0 & 0 & -\alpha_0\overline{\overline{N}} \\ 0 & 0 & \eta & -\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -\theta_0 & 0 \\ -\alpha_2\overline{\overline{T}} & 0 & 0 & 0 & \theta_1 & -\alpha_2\overline{\overline{F}}-\mu_0 \end{bmatrix}$$

Where

$$a_{11} = r - 2\frac{r}{L}\overline{\overline{F}} - \beta_2\overline{\overline{N}} - 2\gamma\overline{\overline{F}}\,\overline{\overline{U}} - \alpha_1\overline{\overline{T}}$$

$$c_{33} = s - 2\frac{s}{k}\overline{\overline{N}} + \pi_2\beta_2\overline{\overline{F}} - \alpha_0\overline{\overline{T}}$$

The stability conclusions from the Jacobian matrix J_4 are not clear at the interior equilibrium $E_4\left(\overline{\overline{F}},0,\overline{\overline{N}},\overline{\overline{P}},\overline{\overline{U}},\overline{\overline{T}}\right)$

6. The stability conclusions from the Jacobian matrix J_5 are not clear at the interior equilibrium $E_5 = \left(F^*,W^*,N^*,P^*,U^*,T^*\right)$,

As a result, Lyapunov's approach has been used to forecast local stability requirements, as shown by the theorem below.

Theorem 4.1. If the following conditions carry, the model is locally asymptotically stable around the equilibrium point E_5:

$(i)\ 4\eta^2\pi_2 k < \lambda_0 s$

$(ii)\ 6k\alpha_2 T^*\alpha_0{}^2 < s\pi_2\alpha_1\left(\alpha_2 F^* + \mu_0\right)$

$(iii)\ 3\beta_1\alpha_2 T^*\delta_1{}^2 W^{*2} < \alpha_1\left(\lambda + \pi_1\beta_1 W^*\right)\left(\alpha_2 F^* + \mu_0\right)\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F^*\right)$

$$(iv)\ \max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0\left(\frac{r}{L}+\gamma U^*\right)}\cdot\frac{9\alpha_1\theta_1{}^2}{\alpha_2 T^*\theta_0\left(\alpha_2 F^* + \mu_0\right)}\right\} < \frac{\theta_0\lambda_0}{6k_1{}^2}$$

(Appendix B contains the proof of this theorem.)

Theorem 4.2. If the following conditions carry, the model is globally asymptotically stable around the equilibrium point E_5:

$(i)\ 4\eta^2\pi_2 k < \lambda_0 s$

$(ii)\ 6k\alpha_2 T^*\alpha_0{}^2 < s\pi_2\alpha_1\left(\alpha_2 F_{\max} + \mu_0\right)$

$(iii)\ 3\beta_1\alpha_2 T^*\delta_1{}^2 W^{*2} < \alpha_1\left(\lambda + \pi_1\beta_1 W^*\right)\left(\alpha_2 F_{\max} + \mu_0\right)\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F_{\max}\right)$

$$(iv)\ \max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0\left(\frac{r}{L}+\gamma U_{\max}\right)}\cdot\frac{9\alpha_1\theta_1{}^2}{\alpha_2 T^*\theta_0\left(\alpha_2 F_{\max} + \mu_0\right)}\right\} < \frac{\theta_0\lambda_0}{6k_1{}^2}$$

Inside the region of attraction Ω of Lemma 3.1

(Appendix C contains the proof of this theorem)

5. NUMERICAL SIMULATION

Moreover, qualitative analysis, the model system (2.1) is also explored quantitatively to anticipate some behavior of the system, with the change in parameter.

For this quantitative analysis and demonstration of the model system, we use MATLAB software package. This model system involving many parameters. So, keeping all obtained qualitative results in mind, the numerical values of these parameters are chosen arbitrarily in the following.

$r = 20$, $L = 100$, $\beta_1 = 0.01$, $\beta_2 = 0.1$, $\gamma = 0.01$, $\alpha_1 = 0.1$, $\lambda = 10$, $\pi_1 = 0.04$, $\delta_1 = 0.5$, $\delta_0 = 0.05$, $s = 5$, $K = 100$, $\pi_2 = 0.02$, $\alpha_0 = 0.002$, $\eta = 0.1$, $\lambda_0 = 0.1$, $Q = 20$, $\kappa_1 = 0.1$, $\theta_0 = 0.01$.(5.1)

The equilibrium point value for the model (2.1) are:

$$F^* = 11.6691, W^* = 7.4537, N^* = 98.9182,$$

$$P^* = 11.0879, U^* = 42.3691, T^* = 33.7060$$

For the model (2.1), the Jacobian matrix eigenvalues that correspond to equilibrium are:

-16.7671, -7.9480, -5.0133, -0.1104, $-0.0002-0.0321i$, $-0.0002+0.0321i$

Here, it may be noted that four eigenvalues of matrix are negative and the other two eigenvalues are with negative real parts. Therefore, it implies that the interior equilibrium is locally asymptotically stable. Now, all the conditions obtained for local stability and global stability are getting satisfied for the above choice of parameters.

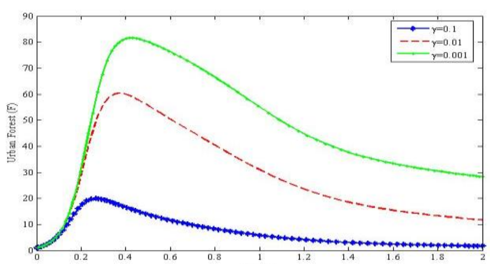

Fig. 1. Variation of density of urban forest territory (F) with time (t) for different values of γ.

Figure 1 shows the variation in urban forest territory with time for various values of, demonstrating that as the value of γ increases, the density of urban forest territory decreases. This implies that the increase in urbanization is also responsible for the depletion of urban forest territory.

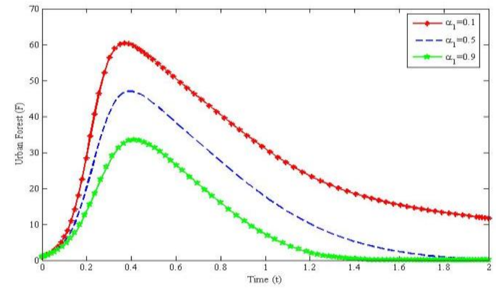

Fig. 2. Variation of density of urban forest territory (F) with time (t) for different values of α_1.

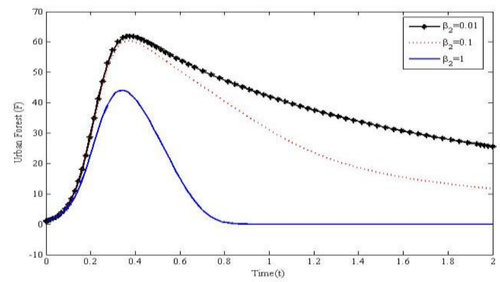

Fig. 3. Variation of density of urban forest territory (F) with time (t) for different values of β_2.

Figure 2 and 3 shows the variation in density of urban forest territory with time for different values of α_1 and β_2 which shows that density of urban forest territory decreases as the values of α_1 and β_2 increases. This indicates that increase of pollutants and human population is responsible for the depletion of urban forest territory.

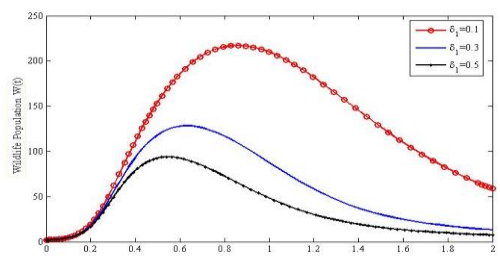

Fig. 4. Variation of density of wildlife population (W) with time (t) for different values of δ_1.

Figure 4 shows the variation in wildlife population density with time for different values of δ_1, demonstrating that as the value of δ_1, increases, density of wildlife population decreases. This means that the increase in pollutant or toxicant is also responsible for the depletion of wildlife population.

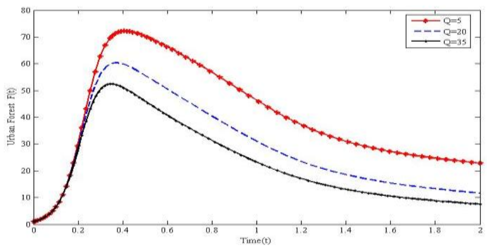

Fig. 5. Variation of density of urban forest territory (F) with time (t) for different values of Q.

Figure 5 shows the variation in density of forest resources with time for different values of Q, which shows that density of forest resources decreases as the values of Q increases.

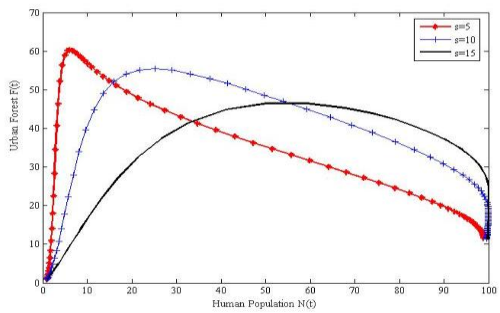

Fig. 6.Variation of density of urban forest territory (F) with human population (N) for different values of s.

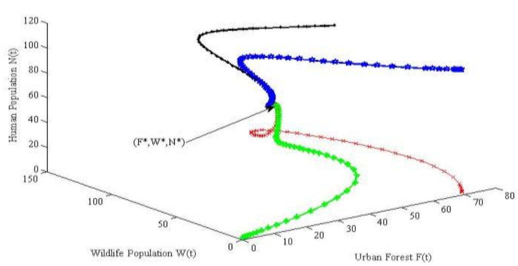

Fig.7. Global stability of interior equilibrium in $F-W-N$

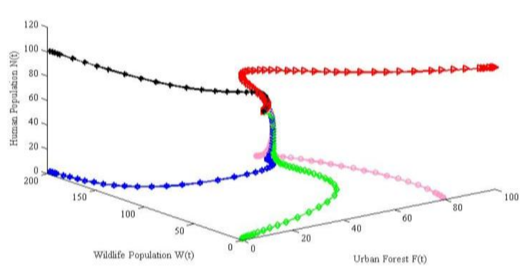

Fig.8. Global stability of interior equilibrium in $F-W-N$.

The Figure 7 and Figure 8 reveals the global stability of interior equilibrium for different initial values of dynamical variable in three-dimensional $F-W-N$ plane with respect to the parameter values defined in Equation (5.1). It demonstrate that the curve reaches to interior equilibrium (F^*, W^*, N^*) with time for the different initial values. A similar pattern of behavior can be observed in other environments.

6. CONCLUSION

In this paper, a non-linear mathematical model has been prepared and explored to study the effects of urban pollution induced by malpractices of urban development in urban forest territory, wildlife population and human population. The proposed model has been constructed using six dynamical variables - urban forest territory, wildlife population, human population, and urban development activities attached with concentration of urban pollution. The obtained model has been solved and equilibrium point has been identified; after which their existence, stabilities have also been explained. The numerical solution and graphical illustrations both of this model have been evaluated for qualitative analysis. As urban development in the urban forest territory grows, the density of urban forest territory and wildlife population decreases. It has also been found that growth of human population puts excessive pressure on forest and also increases the declination rate of forest reserves as well as wildlife population.

This research paper concludes with the view that appropriate balance between urban development and wildlife population has become the foremost requirement in today's scenario. Environment friendly cities is the essential focus area where planning and management should be encouraged to maintain this balance. Various pollution tolerance and mitigation capabilities such as cultivation of trees, shrubs, species, etc. should be adopted to make cities greener and protect the wildlife population in the long run. Urban development activities and apposite utilization of natural resources should go hand in hand, avoiding the malpractices in urban system and making the environment less polluted at the end.

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APPENDIX -A

Lemma 3.1*The set*

$$\Omega = \left\{ \begin{array}{l} (F,\ W,\ N,\ P,\ U,T): 0 \le F \le L, 0 \le W \le W_m, \\ 0 \le N \le N_m, 0 \le P \le P_m, 0 \le U \le U_m, 0 \le T \le T_m \end{array} \right\}$$

Where, $W_m = \dfrac{\lambda L}{\delta_0 - \pi_1 \beta_1 L}$ with $\delta_0 - \pi_1 \beta_1 L > 0$,

$N_m = \frac{k}{s}\left(s + \pi_2\beta_2 L - \alpha_0 T_m\right)$ with

$\left(s + \pi_2\beta_2 L - \alpha_0 T_m\right) > 0$, $P_m = \frac{\eta N_m}{\lambda_0}$,

$U_m = \frac{Q + k_1 P_m}{\theta_0}$, $T_m = \frac{Q_0 + \theta_1 U_m}{\mu_0}$

Proof. By applying comparison theorem in the model (2.1), as of the Equation (2.1a) of the model (2.1) provides

$$\frac{dF}{dt} \le rF\left(1 - \frac{F}{L}\right)$$

This provides $0 \le F \le L$

As of the Equation (2.1b) of the model (2.1) provides

$$\frac{dW}{dt} \le \lambda F + \pi_1\beta_1 FW - \delta_0 W$$

This gives

$$0 \le W \le \frac{\lambda L}{\delta_0 - \pi_1\beta_1 L} = W_m$$

with $\delta_0 - \pi_1\beta_1 L > 0$

So $0 \le W \le W_m$

As of the Equation (2.1c) of the model (2.1) provides

$$\frac{dN}{dt} \le sN\left(1 - \frac{N}{K}\right) + \pi_2\beta_2 FN - \alpha_0 NT$$

This provides

$$0 \le N \le \frac{k}{s}\left(s + \pi_2\beta_2 L - \alpha_0 T\right) = N_m$$

$0 \le N \le N_m$ with $s + \pi_2\beta_2 L - \alpha_0 T > 0$

As of the Equation (2.1d) of the model (2.1) provides

$$\frac{dP}{dt} \le \lambda N - \lambda_0 P$$

This provides

$$0 \le P \le \frac{\lambda}{\lambda} N_m = P_m$$

$$0 \le P \le P_m$$

As of the Equation (2.1e) of the model (2.1) provides

$$\frac{dU}{dt} \le Q + k_1 P - \theta_0 U$$

This provides

$$0 \le U \le \frac{Q + k_1 P_m}{\theta_0} = U_m$$

$$0 \le U \le U_m$$

As of the Equation (2.1f) of the model (2.1) provides

$$\frac{dT}{dt} \le Q_0 + \theta_1 U - \mu_0 T$$

This provides

$$0 \le T \le \frac{Q_0 + \theta_1 U_m}{\mu_0} = T_m$$

$$0 \le T \le T_m$$

Hence, the lemma follows.

APPENDIX -B

Theorem 4.1: If the following conditions carry, the model is locally asymptotically stable around the equilibrium point:

(i) $4\eta^2\pi_2 k < \lambda_0 s$

(ii) $6k\alpha_2 T^*\alpha_0^{\,2} < s\pi_2\alpha_1\left(\alpha_2 F^* + \mu_0\right)$

(iii) $3\beta_1\alpha_2 T^*\delta_1^{\,2} W^{*2} < \alpha_1\left(\lambda + \pi_1\beta_1 W^*\right)\left(\alpha_2 F^* + \mu_0\right)\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F*\right)$

(iv)

$$\max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0 \frac{r}{L} + \gamma U^*} . \frac{9\alpha_1\theta_1^{\,2}}{\alpha_2 T^*\theta_0\left(\alpha_2 F^* + \mu_0\right)}\right\} < \frac{\theta_0\lambda_0}{6k_1^{\,2}}$$

Proof. Implement the subsequent conversions around E_5 to prove Theorem:

$F = F^* + F_1, W = W^* + W_1, N = N^* + N_1, P = P^* + P_1, U = U^* + U_1$ and $T = T^* + T_1$

where F_1, W_1, N_1, P_1, U_1 and T_1 are minor perturbations about $E_5\left(F^*, W^*, N^*, P^*, U^*, T^*\right)$,

Applying the positive definite function as an example,

$$V = \frac{1}{2}\left(\frac{F_1^{\,2}}{F^*} + l_1 W_1^{\,2} + l_2\frac{N_1^{\,2}}{N^*} + l_3 P_1^{\,2} + l_4 U_1^{\,2} + l_5 T_1^{\,2}\right) \quad \text{(B.1)}$$

where l_1, l_2, l_3, l_4 and l_5 are positive constants.

After taking derivative of the Equation (B.1) with relative to t, we attain

$$\frac{dV}{dt}=\frac{F_1}{F^*}\frac{dF_1}{dt}+l_1W_1\frac{dW_1}{dt}+l_2\frac{N_1}{N^*}\frac{dN_1}{dt}+l_3P_1\frac{dP_1}{dt}+l_4U_1\frac{dU_1}{dt}+l_5T_1\frac{dT_1}{dt} \quad \text{(B.2)}$$

Now, putting all equations of the model system (2.1), Equation (B.2) shrinks to

$$\frac{dV}{dt}=-\left(\frac{r}{L}+\gamma U^*\right)F_1^{\,2}-l_1\left(\delta_0+\delta_1T^*-\pi_1\beta_1F^*\right)W_1^{\,2}$$
$$-l_2\frac{s}{k}N_1^{\,2}-l_3\lambda_0P_1^{\,2}-l_4\theta_0U_1^{\,2}-l_5\left(\alpha_2F^*+\mu_0\right)T_1^{\,2}+$$
$$\left(-\beta_1+l_1\lambda+l_1\pi_1\beta_1W^*\right)F_1W_1+\left(-\beta_2+l_2\pi_2\beta_2\right)F_1N_1+$$
$$+\left(-\alpha_1+l_5\alpha_2T^*\right)F_1T_1-\gamma F^*F_1U_1-$$
$$l_1\delta_1W^*W_1T_1-l_2\alpha_0T_1N_1+l_3\eta P_1N_1+l_4k_1P_1U_1+l_5\theta_1U_1T_1 \quad \text{(B.3)}$$

Now if we choose

$$-\beta_1+l_1\left(\lambda+\pi_1\beta_1W^*\right)=0$$

$$l_1=\frac{\beta_1}{\lambda+\pi_1\beta_1W^*}$$ which is positive constant.

And, $-\beta_2+l_2\pi_2\beta_2=0$

$$l_2=\frac{1}{\pi_2}$$ (a positive constant)

Also, $-\alpha_1+l_5\alpha_2T^*=0$

$$l_5=\frac{\alpha_1}{\alpha_2T^*}$$ (a positive constant)

Now, we have

$$\frac{dV}{dt}=-\left(\frac{r}{L}+\gamma U^*\right)F_1^{\,2}-\gamma F^*F_1U_1-\frac{1}{3}l_4\theta_0U_1^{\,2}$$
$$-\frac{1}{3}l_4\theta_0U_1^{\,2}+l_4k_1P_1U_1-\frac{1}{2}l_3\lambda_0P_1^{\,2}$$
$$-\frac{1}{2}l_3\lambda_0P_1^{\,2}+l_3\eta P_1N_1-\frac{1}{2}l_2\frac{s}{k}N_1^{\,2}$$
$$-\frac{1}{2}l_2\frac{s}{k}N_1^{\,2}-l_2\alpha_0T_1N_1-\frac{1}{3}l_5\left(\alpha_2F^*+\mu_0\right)T_1^{\,2}$$
$$-\frac{1}{3}l_5\left(\alpha_2F^*+\mu_0\right)T_1^{\,2}+l_5\theta_1U_1T_1-\frac{1}{3}l_4\theta_0U_1^{\,2}$$
$$-\frac{1}{3}l_5\left(\alpha_2F^*+\mu_0\right)T_1^{\,2}-l_1\delta_1W^*W_1T_1-$$
$$l_1\left(\delta_0+\delta_1T^*-\pi_1\beta_1F^*\right)W_1^{\,2}$$

Now, $\frac{dV}{dt}$ will be negative if

1. $$\left(\gamma F^*\right)^2<\frac{1}{3}\left(\frac{r}{L}+\gamma U^*\right)l_4\theta_0$$

which can be written as

$$\frac{3\gamma^2F^{*2}}{\left(\frac{r}{L}+\gamma U^*\right)\theta_0}<l_4 \quad \text{(B.5)}$$

2. $$\left(l_4k_1\right)^2<\frac{1}{3}l_4\theta_0\frac{1}{2}l_3\lambda_0$$

Here, if we choose $l_3=1$, we get

$$l_4<\frac{\theta_0\lambda_0}{6k_1^{\,2}} \quad \text{(B.6)}$$

3. $$\left(l_3\eta\right)^2<\frac{1}{2}l_3\lambda_0\frac{1}{2}l_2\frac{s}{k}$$

which can be written as

$$4\eta^2\pi_2k<\lambda_0s$$

4. $$\left(l_2\alpha_0\right)^2<\frac{1}{2}l_2\frac{s}{k}\frac{1}{3}l_5\left(\alpha_2F^*+\mu_0\right)$$

which can be written as

$$6k\alpha_2T^*\alpha_0^{\,2}<s\pi_2\alpha_1\left(\alpha_2F^*+\mu_0\right) \quad \text{(B.7)}$$

5. $$\left(l_5\theta_1\right)^2<\frac{1}{3}l_5\left(\alpha_2F^*+\mu_0\right)\frac{1}{3}l_4\theta_0$$

which can be written as

$$\frac{9\alpha_1\theta_1^{\,2}}{\alpha_2 T^*\left(\alpha_2 F^* + \mu_0\right)\theta_0} < l_4 \qquad \text{(B.9)}$$

6. $\left(l_1\delta_1 W^*\right)^2 < \frac{1}{3} l_5\left(\alpha_2 F^* + \mu_0\right)$
$l_1\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F^*\right)$

which can be written as

$$3\beta_1\alpha_2 T^*\delta_1^{\,2} W^{*2} < \alpha_1\left(\lambda + \pi_1\beta_1 W^*\right)$$
$$\left(\alpha_2 F^* + \mu_0\right)\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F*\right)$$

(B.10)

Also,

$$\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F^*\right) > 0 \qquad \text{(B.11)}$$

As of Equations (B.5), (B.6) and (B.9), we achieve

$$\max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0\left(\frac{r}{L} + \gamma U^*\right)} \cdot \frac{9\alpha_1\theta_1^{\,2}}{\alpha_2 T^*\theta_0\left(\alpha_2 F^* + \mu_0\right)}\right\} < l_4 < \frac{\theta_0\lambda_0}{6k_1^{\,2}}$$

(B.12)

which ultimately reduces to

$$\max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0\left(\frac{r}{L} + \gamma U^*\right)} \cdot \frac{9\alpha_1\theta_1^{\,2}}{\alpha_2 T^*\theta_0\left(\alpha_2 F^* + \mu_0\right)}\right\} < \frac{\theta_0\lambda_0}{6k_1^{\,2}}$$

(B.13)

The model system demonstrates local asymptotic stable behavior for the obtained conditions (B.7), (B.8), (B.10), (B.11), and (B.13).

APPENDIX- C

Theorem 4.2. If the following conditions carry, the model is globally asymptotically stable around the equilibrium point E_5:

(i) $4\eta^2\pi_2 k < \lambda_0 s$

(ii) $6k\alpha_2 T^*\alpha_0^{\,2} < s\pi_2\alpha_1\left(\alpha_2 F_{\max} + \mu_0\right)$

(iii) $3\beta_1\alpha_2 T^*\delta_1^{\,2} W^{*2} < \alpha_1\left(\lambda + \pi_1\beta_1 W^*\right)$
$\left(\alpha_2 F_{\max} + \mu_0\right)\left(\delta_0 + \delta_1 T^* - \pi_1\beta_1 F_{\max}\right)$

(iv) $\max\left\{\frac{3\gamma^2 F^{*2}}{\theta_0\left(\frac{r}{L} + \gamma U^*\right)} \cdot \frac{9\alpha_1\theta_1^{\,2}}{\alpha_2 T^*\theta_0\left(\alpha_2 F^* + \mu_0\right)}\right\} < \frac{\theta_0\lambda_0}{6k_1^{\,2}}$

Proof. Used to verify the subsequent positive definite function to prove the theorem:

$$B = \left(F - F^* - F^*\ln\frac{F}{F^*}\right) + \frac{1}{2}m_1\left(W - W^*\right)^2$$
$$+ m_2\left(N - N^* - N^*\ln\frac{N}{N^*}\right) + \frac{1}{2}m_3\left(P - P^*\right)^2 +$$
$$\frac{1}{2}m_4\left(U - U^*\right)^2 + \frac{1}{2}m_5\left(T - T^*\right)^2$$

(C.1)

where m_1, m_2, m_3, m_4 and m_5 are positive constants.

After differentiating of Equation (C.1), with relative to t, we get

$$\frac{dB}{dt} = \frac{\left(F - F^*\right)}{F}\frac{dF}{dt} + m_1\left(W - W^*\right)\frac{dW}{dt}$$
$$+ m_2\frac{\left(N - N^*\right)}{N}\frac{dN}{dt} + m_3\left(P - P^*\right)\frac{dP}{dt} +$$
$$+ m_4\left(U - U^*\right)\frac{dU}{dt} + m_5\left(T - T^*\right)\frac{dU}{dt}$$

(C.2)

Substituting all equations of the model (2.1), Equation (C.2) gives

$$\frac{dB}{dt} = -\left(\frac{r}{L} + \gamma U\right)\left(F - F^*\right)^2 - \beta_1\left(F - F^*\right)\left(W - W^*\right) - \beta_2\left(F - F^*\right)\left(N - N^*\right)$$
$$-\gamma F^*\left(F - F^*\right)\left(U - U^*\right) - \alpha_1\left(F - F^*\right)\left(T - T^*\right) + m_1\lambda\left(F - F^*\right)\left(W - W^*\right)$$
$$+ m_1\left(W - W^*\right)\left[\pi_1\beta_1\left\{F\left(W - W^*\right) + W^*\left(F - F^*\right)\right\}\right] - m_1\delta_1\left(W - W^*\right)\left\{T\left(W - W^*\right) + W^*\left(T - T^*\right)\right\}$$
$$-m_1\delta_0\left(W - W^*\right)^2 + m_2\left(N - N^*\right)\left\{-\frac{s}{k}\left(N - N^*\right) + \pi_2\beta_2\left(F - F^*\right) - \alpha_0\left(T - T^*\right)\right\}$$
$$m_3\left(P - P^*\right)\left\{\eta\left(N - N^*\right) - \lambda_0\left(P - P^*\right)\right\} + m_4\left(U - U^*\right)\left\{k_1\left(P - P^*\right) - \theta_0\left(U - U^*\right)\right\}$$
$$+ m_5\left(T - T^*\right)\left\{\theta_1\left(U - U^*\right) - \alpha_2\left(FT - F^*T^*\right) - \mu_0\left(T - T^*\right)\right\}$$

$$\begin{aligned}\frac{dB}{dt} &= -\left(\frac{r}{L}+\gamma U\right)\left(F-F^*\right)^2+\left(-\beta_1+m_1\lambda+m_1\pi_1\beta_1W^*\right)\left(F-F^*\right)\left(W-W^*\right)\\ &+\left(\beta_2+m_2\pi_2\beta_2\right)\left(F-F^*\right)\left(N-N^*\right)+\left(-\alpha_1+m_5\alpha_2T^*\right)\left(F-F^*\right)\left(T-T^*\right)\\ &-\left(\delta_0+\delta_1T-m_1\pi_1\beta_1F\right)\left(W-W^*\right)^2-m_2\frac{s}{k}\left(N-N^*\right)^2-m_3\lambda_0\left(P-P^*\right)^2\\ &-m_4\theta_0\left(U-U^*\right)^2-m_5\left(\alpha_2F+\mu_0\right)\left(T-T^*\right)^2-\gamma F^*\left(F-F^*\right)\left(U-U^*\right)^2\\ &-m_1\delta_1W^*\left(W-W^*\right)\left(T-T^*\right)-m_2\alpha_0\left(N-N^*\right)\left(T-T^*\right)\\ &+m_3\eta\left(P-P^*\right)\left(N-N^*\right)+m_4k_1\left(P-P^*\right)\left(U-U^*\right)+m_5\theta_1\left(U-U^*\right)\left(T-T^*\right)\end{aligned} \quad \text{(C.3)}$$

Now, if we choose

$$-\beta_1+m_1\left(\lambda+\pi_1\beta_1W^*\right)=0$$

$$m_1=\frac{\beta_1}{\lambda+\pi_1\beta_1W^*}$$ which is a positive constant.

$$-\beta_2+m_2\beta_2\pi_2=0$$

$$m_2=\frac{1}{\pi_2}$$ which is a positive constant.

$$-\alpha_1+m_5\alpha_2T^*=0$$

$$m_5=\frac{\alpha_1}{\alpha_2T^*}$$ which is also a positive constant.

Now, reduces Equation (C.3) to

$$\begin{aligned}\frac{dB}{dt} &= -\left(\frac{r}{L}+\gamma U\right)\left(F-F^*\right)^2-m_1\left(\delta_0+\delta_1T-m_1\pi_1\beta_1F\right)\left(W-W^*\right)^2-m_2\frac{s}{k}\left(N-N^*\right)^2\\ &-m_3\lambda_0\left(P-P^*\right)^2-m_4\theta_0\left(U-U^*\right)^2-m_5\left(\alpha_2F+\mu_0\right)\left(T-T^*\right)^2-\gamma F^*\left(F-F\quad\right)\left(U-U^*\right)\\ &-m_1\delta_1W^*\left(W-W^*\right)\left(T-T^*\right)-m_2\alpha_0\left(N-N^*\right)\left(T-T^*\right)+m_3\eta\left(P-P^*\right)\left(N-N^*\right)\\ &+m_4k_1\left(P-P^*\right)\left(U-U^\circ\right)+m_5\theta_1\left(U-U^*\right)\left(T-T^*\right)\end{aligned}$$

$$\begin{aligned}\frac{dB}{dt} &= -\left(\frac{r}{L}+\gamma U\right)\left(F-F^*\right)^2-\gamma F^*\left(F-F^*\right)\left(U-U^*\right)-\frac{1}{3}m_4\theta_0\left(U-U^*\right)^2\\ &-\frac{1}{3}m_4\theta_0\left(U-U^*\right)^2+m_4k_1\left(P-P^*\right)\left(U-U^*\right)-\frac{1}{2}m_3\lambda_0\left(P-P^*\right)^2\\ &-\frac{1}{2}m_3\lambda_0\left(P-P^*\right)^2+m_3\eta\left(P-P^*\right)\left(N-N^*\right)-\frac{1}{2}m_2\frac{s}{k}\left(N-N^*\right)^2\\ &-\frac{1}{2}m_2\frac{s}{k}\left(N-N^*\right)^2-m_2\alpha_0\left(N-N^*\right)\left(T-T^*\right)-\frac{1}{3}m_5\left(\alpha_2F+\mu_0\right)\left(T-T^*\right)^2\\ &-\frac{1}{3}m_5\left(\alpha_2F+\mu_0\right)\left(T-T^*\right)^2+m_5\theta_1\left(U-U^*\right)\left(T-T^*\right)-\frac{1}{3}m_4\theta_0\left(U-U^*\right)^2\\ &-\frac{1}{3}m_5\left(\alpha_2F+\mu_0\right)\left(T-T^*\right)^2-m_1\delta_1W^*\left(W-W^*\right)\left(T-T^*\right)-m_1\left(\delta_0+\delta_1T-\pi_1\beta_1F\right)\left(W-W^*\right)^2\end{aligned} \quad \text{(C.4)}$$

Now, $\frac{dB}{dt}$ will be negative if

(i) $$\left(\gamma F^*\right)^2<\frac{1}{3}\left(\frac{r}{L}+\gamma U_{\max}\right)m_4\theta_0$$

or

$$\frac{3\gamma^2 F^{*2}}{\left(\frac{r}{L}+\gamma U_{\max}\right)\theta_0} < m_4 \qquad \text{(C.5)}$$

4 $\left(m_4 k_1\right)^2 < \frac{1}{3} m_4 \theta_0 \frac{1}{2} m_3 \lambda_0$

Or $m_4 < \frac{\theta_0 \lambda_0}{6k_1^2}$ (C.6)

5 $\left(m_3 \eta\right)^2 < \frac{1}{2} m_3 \lambda_0 \frac{1}{2} m_2 \frac{s}{k}$

Here, if we choose $m_3 = 1$, then we have

$$4\eta^2 \pi_2 k < \lambda_0 s \qquad \text{(C.7)}$$

6 $\left(m_2 \alpha_0\right)^2 < \frac{1}{2} m_2 \frac{s}{k} \frac{1}{3} m_5 \left(\alpha_2 F_{\max} + \mu_0\right)$

which can be written as

$$6k\alpha_2 T^* \alpha_0^2 < s\pi_2 \alpha_1 \left(\alpha_2 F_{\max} + \mu_0\right) \qquad \text{(C.8)}$$

7 $\left(m_5 \theta_1\right)^2 < \frac{1}{3} m_5 \left(\alpha_2 F_{\max} + \mu_0\right) \frac{1}{3} m_4 \theta_0$

or

$$\frac{9\alpha_1 \theta_1^2}{\alpha_2 T^* \left(\alpha_2 F_{\max} + \mu_0\right)\theta_0} < m_4 \qquad \text{(C.9)}$$

8 $\left(m_1 \delta_1 W^*\right)^2 < \frac{1}{3} m_5 \left(\alpha_2 F_{\max} + \mu_0\right)$ or

$m_1 \left(\delta_0 + \delta_1 T_{\max} - \pi_1 \beta_1 F_{\max}\right)$

$$3\beta_1 \alpha_2 T^\circ \delta_1^2 W^{*2} < \alpha_1 \left(\lambda + \pi_1 \beta_1 W^*\right) \left(\alpha_2 F_{\max} + \mu_0\right)\left(\delta_0 + \delta_1 T_{\max} - \pi_1 \beta_1 F_{\max}\right) \qquad \text{(C.10)}$$

or

$$\delta_0 + \delta_1 T_{\max} - \pi_1 \beta_1 F_{\max} > 0 \qquad \text{(C.11)}$$

From Equation (C.5), (C.6) and (C.9), we get

$$\max\left\{\frac{3\gamma^2 F^{*2}}{\left(\frac{r}{L}+\gamma U_{\max}\right)\theta_0} . \frac{9\alpha_1 \theta_1^2}{\alpha_2 T^* \left(\alpha_2 F_{\max} + \mu_0\right)\theta_0}\right\} < m_4 < \frac{\theta_0 \lambda_0}{6k_1^2} \qquad \text{(C.12)}$$

which ultimately gives

$$\max\left\{\frac{3\gamma^2 F^{*2}}{\left(\frac{r}{L}+\gamma U_{\max}\right)\theta_0} . \frac{9\alpha_1 \theta_1^2}{\alpha_2 T^* \left(\alpha_2 F_{\max} + \mu_0\right)\theta_0}\right\} < \frac{\theta_0 \lambda_0}{6k_1^2} \qquad \text{(C.13)}$$

The above derived conditions give the global asymptotic stable behavior.