Out of Infinitesimals and Limits: Towards A Reinstitution of Mathematical Analysis of Finite Quantities

I.S.Patil [M.A]
Mathematical Analysis, K.K.Koppa

***Abstract* - As the title of this paper indicates, it is argued in this paper that the derivative of the function is obtainable without the doctrine of infinitesimals and limits. When Newton did it without them why not we?**

INTRODUCTION

The difference quotient of [f(x+h)-f(x)/h)] is considered to be the derivative of the function. To one's astonishment, this differential quotient, after having been deduced from the above formula, is not exactly same to the anticipated derivative. It needs some approximations and manipulations.

That means the result of f(x+h)-f(x)/h is not exactly the derivative of the function! The definition of derivative is not the definition of the same derivative! A contradiction to start with! The derivative of the function deduced from the above formula is always greater than f'(x)-epsilon and less than f'(x)+epsilon. And f'(x) is considered the differential or derivative or dy/dx of f(x)! The definition of derivative is the difference quotient of f(x+h)-f(x)/h. But, in fact it is not the derivative exactly. Epsilon symbol represents errors reduced less than sensible. That is, beyond the senses. But it is not possible to reduce it less than or beyond reason. The conjecture of infinitesimals and limits would have been accepted were there a possibility of obtaining the same results for the increments irrespective of magnitudes. When we found that the method is not objective but manipulative, is not acceptable. This is tricky and vague. An admitted and frequently used formula to find the differential or derivative of a function i.e, lim f'(x) = f(x+h) – f(x)/h, as h→0, is not equal to f'(x), but falls short of that. There seems to be a logical fallacy. To begin with, first we define f'(x) is the difference quotient of f(x+h)-f(x)/h, and then we observe that this differential qoutient is not exactly the derivative of the function. Reconciling the inconsistent principles or basic contradiction is taken up by mathematicians since more than three centuries but in vain. Thus, the formula, a symbolic form of conceptual tradition, in its final stage or climax fails to deliver truth.

The idea of differential quotient has its origin in the Axioms of equality, equal increase and equal decrease of variables in an equation, according to **Euclid**. In any equation y = f(x) or y = x it is supposed that an increment in y and an increment in x do not imbalance the equation, i.e, the variable quantities increase synchronously. That means symbolically y+dy = x+dx or in modern terms y+dy = f(x+dx). Finding out the equivalent increment of dy is done with simple operation by subtracting equal terms y and f(x) from both sides of that increased equation. If y+dy =f(x+dx) then dy must be equal to f(x+dx)-f(x). According to our supposition of synchronous increments, the remainder of the whole equation, following the subtraction of equals y and f(x) must remain equal. Let us have a look:

dy = f(x+dx) – y or
dy = f(x+dx) – f(x). [dx and h are used for the same quantity, a small increment)

This is true and accurate in cases of y = f(x) where f(x) is a product of a constant and a variable.

For example: y = f(x) = 2x
y+dy = f(x+dx) = 2(x+dx)
y+dy = 2x+2dx
dy = 2x+2dx-y
= 2dx.

And f(dx)= 2dx, or dy = 2dx., and dy/dx=2.

But the application of this supposition fails to lead to truth if applied to other kind of problems involving product of two variable quantities or to curves.

Example: $y = f(x) = x^2$
$y+dy = f(x+dx) = (x+dx)^2$
$y+dy = f(x+dx) = x^2+2xdx+dx^2$
$dy = f(x+dx)-f(x) = x^2+2xdx+dx^2-y$
$dy = 2xdx+dx^2$.and $dy/dx=2x+dx$.

It is very true that the difference between x^2 and its next increased quantity $(x+dx)^2$ is $2xdx+dx^2$. And the ratio of the difference of dy and dx or dy/dx = $2xdx+dx^2/dx$. Therefore dy/dx = 2x+dx.
But 2x+dx is not the true and precise result. Though it is obtained in a manner nothing unscientific.

From Sir Isaac Newton's method of fluxions, we could deduce that the fluxion of x^2 is only 2x. Newton's method gives us not the difference between two successive quantities, rather it gives us the ratio of change even when it is unchanged. It is called instantaneous rate of change. This instantaneous rate of change of x^2 is only 2x.
And 2x is the true result.
Thus there arises a million dollar question, how is it that the scientifically derived ratio of increments leads to inaccurate conclusion 2x+dx? Or why 2x+dx is the false conclusion?
Instead of making thorough investigation in the matter, mathematicians simply overlooked that perplexity and concentrated on efforts to reduce 2x+dx to 2x. They wanted to reconcile two irreconcilable quantities. That is done by introducing the concept 'infinitesimals'. With the help of this concept and a symbol $dx \rightarrow 0$, dx out of 2x+dx is arbitrarily taken away and only 2x is retained. This looks like an attempt of wishful thinking, not an objective science. Because, 'infinitesimals' is a strange concept, from which we can prove anything. The concept suffers from inherent contradiction. Infinitesimal is a quantity infinitely diminished or reduced. At the same time it is not a quantity of finite magnitude. The quantity which is neither zero nor of any magnitude is impossible to be conceived. This seems to be a kind of clear obscurity of the concept. It is denoted by the symbol $dx \rightarrow 0$. $dx \rightarrow 0$ means, $dx \neq$ zero, $dx \neq$ finite quantity.
Thus one may infer that if x+dx = x.

Or x-dx = x

Then dx = x-x = zero.

It is plain to every one that dx is nothing or zero.
Surprisingly, from this concept the derivation of x^2 is obtained as below:

$y = x^2$

$y+dy = (x+dx)^{2.}$

$dy = x^2+2xdx+dx^2-y$

$= 2xdx+dx^2$

$dy/dx = 2xdx+dx^2/dx = 2x+dx$

At this stage dx is treated as zero by the notation $dx \rightarrow 0$, and obtain dy /dx=2x.
If dy/dx = 2x on the presumption of dx = 0.
Then dy/0 = 2x

$dy = 2x \times 0 = 0.$

Thus dy/dx model leads us to nowhere.
In order to save dy/dx operational concept from its dubious character one more complimentary concept limit is introduced. According to this concept the conclusion 2x+dx is the result, the limit of which is only 2x, when the limit of dx is zero. Limit of dx is zero means, dx never arrives at zero but very close to zero. How much close? not answerable.
If dx is not zero, then it must be of a finite quantity. they say that dx is a quantity diminished in infinitum. It has no measurable magnitude. It cannot be further divided and at the same time it is not zero. It strains our senses to frame any idea of such quantity. Therefore **George Berkely** asked in fury, whether it be a 'ghost of the departed quantity'.

Thus limits and infinitesimals are unwanted consequences of our intended results. They are not natural or rational or objective conjectures. Infinitesimals and limits may become obsolete once it is possible to derive fluxions or rate of change accurately. This can be done only by instituting principles of analysis of finite quantities.

Two errors and truth:
George Berkely (1685-1753) who carefully investigated into the matter observed that there are two errors in the method or procedure of finding derivative of the function, compensating each other. One error is that the truth comes out more, that means the procedure must be erroneous, and the other one is an attempt to do away with that excess quantity without following rules of logic. According to Berkely this cannot be called science of principles, but science of conclusion.
The first error Berkely noticed that, for example, in an equation $y = x^2$, it is customarily reached to the point of $dy=2xdx+dx^2$. and dy/dx =2xdx+dx.
This exceeds truth, for the truth is only 2x. And this is the first error.
Then this excessive quantity appeared in dy is destroyed wishfully, by using the concept of infinitesimals. This is the second error. And the truth

$dy = 2xdx$ or $dy/dx = 2x$ is arrived at by arbitrary manipulations, and certainly not by scientific method. $dy = 2xdx+dx^2$ is wrong. Why?

In finding the fluxion of x^2 we mean to find the ratio of x to x^2. When x flows uniformly, in what proportion f(x) flows or increases?

In order to find this proportion, we take an increment to x and by such uniform increment x becomes x+1. At the same time x^2 becomes $(x+1)^2$. i.e. x^2+2x+1.

The ratio of increment of x to that of x^2 is obtained by deducting x^2 from $(x+1)^2$.

$$x^2+2x+1-x^2 = 2x+1.$$

Thus the ratio of increment to x and x^2 are 1 to 2x+1.

We expect 1=2x only, or 1:2x. Is it possible to discard 1 out of 2x+1 wishfully without any valid reason?

Where are we deceived?

Instead of 1, if dx is considered as an increment, then also, we arrive at the same result. $2xdx+dx^2$, in the place of 2x+1, which gives on simplification 2x+1 only.

Where the mystery lies? Let us try to find the momentum or differential of rectangle formed by two indeterminate quantities x and y. We suppose that the increments of x and y are dx and dy respectively. The increased rectangle due to the increased sides dx and dy is xy+ydx+xdy+dxdy. The difference due to the increments is obtainable by subtracting the former rectangle xy from the increased rectangle (x+dx) (y+dy). Again dxdy emerges as an excess quantity. Leibnizians supposed the increments dy and dx are infinitesimals and the product of dxdy is also infinitesimal and therefore rejectionable. But they do not reject xdy and ydx which is very strange. A quantity multiplied by infinitesimal must also be rejectionable. But they do not reject xdy and ydx. And consider xdy+ydx as a true differential of rectangle xy. Not only to Berkely but to all thinking men this reasoning appears unfair and unscientific. The true momentum of rectangle xy may be precisely obtained by a clear method. How this can be done by the application of commutative property, I will explain it later. But here I have to make it clear what derivative means?

It can be clearly seen that the binomial of power 1 has no derivative or fluxion of either term. For example, $(a+b)^1 = a^1+b^1 = a+b$. no derivative of either a or b.

But as the power increases or decreases, then there are fluxions of either terms a and b.

Example: $(a+b)^2 = a^2+2ab+b^2$.

2ab is the fluxion of a^2 and b^2, by differentiating i.e. $2ab/b = 2a$ which is the fluxion of a^2.

In the same way the fluxion of b^2 can be obtained by differentiation, $2ab/a = 2b$.

The same reasoning may be applied to other powers.

Example: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

$3a^2b+3ab^2$ are fluxions of a^3 and b^3.

Fluxion of $a^3 = 3a^2b/b = 3a^2$.

Fluxion of $b^3 = 3ab^2/a = 3b^2$. Now we are able to derive the derivative of nth power of any variable, second, third, fourth.... orders of derivatives. I will discuss this in the next article.

And I hope this conceptual shift from difference to fluxion may lead to do away with infinitesimals and limits, without which Newton arrived at true results.