Connectedness and Cut-Point in Topological Space

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***Abstract* - In this paper, we study connectedness in reference to cut-points leading to the introduction of cut-point space. We establish some important results regarding cut-point.**

***Index Terms* - Connectedness, disconnectedness, separated sets.**

INTRODUCTION

For basic definitions and terminology, one may refer to [1-3], where many more references may be found.

Cut- Point: - Let X be a connected topological space. A point x ∈ X is said to be a cut point of X if X – {x} is a disconnected subset of X, i.e., if there is a A|B separation of X – {x} in X -{x}.
In other words,
if X - {x} =$A \cup B$ where $A \neq \emptyset$, B ≠ $\emptyset$
$\overline{A} \cap B$ =ϕ =A ∩ $\overline{B}$ where $\overline{A}$, $\overline{B}$ are the closure of A, B with respect to the topology of X then x is called a cut point.
A point which is not a cut point of X is called a non-cut- point.

Remark: Let X be a topological space and a point x ∈ X is said to be closed if {x} is closed subset of X. A point x ∈ X is said to be open if {x} is open subset of X.

Cut – point space: A non- empty connected topological space X is said to be a cut- point space if every x in X is a cut –point of X.

Example: Let ℝ be the set of real number with usual topology u.
For each x ∈ ℝ
ℝ - {x} = (−∞, x)∪ (x, ∞)
where (− ∞, X), (X, ∞) are open sets in ℝ.
Therefore, x is a cut-point of ℝ
Thus, every point of ℝ is a cut point, and so ℝ is a cut point space.

Example: Let Z denotes that the set of all integers. We define a topology on Z as follows:
Let x ∈ X
We define $W_x = \begin{cases} \{x\} & if\ x\ is\ even \\ \{x-1, x, x+1\} & if\ x\ is\ odd \end{cases}$
Let $\beta = \{ W_x : x \in Z\}$
Clearly $\cup_{x \in Z} W_x$ = Z
Let x, y ∈ Z, $x \neq y$.
Case (i) x and y are even,
W_x= {x},W_y = {y}
$W_x \cap W_y = \phi$

Case (ii)x is even and y is odd
We suppose that $W_x \cap W_y \neq \phi$
Let z ∈ $W_x \cap W_y$
Then z ∈ W_x, z ∈ W_y
Since x is even, soW_x= {x}.
Therefore, z = x
Thus W_z = {z} = {x} i.e., z ∈ $W_z \subset W_x \cap W_y$.

Case (iii) x and y are odd.
Suppose that $W_x \cap W_y \neq \phi$
Let z∈ $W_x \cap W_y$
Now W_x = {x-1, x, x+1}
W_y = {y-1, y, y+1}
Since $W_x \cap W_y \neq \emptyset$, so either x-1 = y-1 or x-1 = y or x-1 = y + 1
Or x = y-1 or x = y or x = y+1 or x + 1 = y-1 or x +1 = y or x +1 = y + 1
Since x ≠ y and both x, y is odd, so we have
Either x-1 = y + 1 or x + 1 = y – 1 i.e., either z = x-1 = y+1
Or z = x + 1 = y – 1
This shows that z is even
So W_z = {z}
Thus, in this case z ∈ $W_z \subset W_x \cap W_y$
From these cases, we see that β become a base for a topology say τ_β on Z. Hence (Z, τ_β) is a topological space. Further, we suppose that (Z, τ_β) is connected.

Suppose Z = A ∪B where A and B are two disjoint non – empty open sets.
Let a $\in A$, $b \in B$
Without loss of generality, we suppose that a < b.
Also, we suppose that a = $a_0 < a_1 < \cdots \ldots \ldots \ldots \; a_{n-1} < a_n$= b in Z.
From this we see that $a_m \in A$, $a_{m+1} \in B$ for some $0 \leq m \leq n-1$
But $W_{a_m} \cap W_{a_{m+1}} \neq \phi$
Since A and B are both open in Z, so $W_{a_m} \subset$ A and $W_{a_{m+1}} \subset B$
As A $\cap B = \phi$, so $W_{a_m} \cap W_{a_{m+1}} = \emptyset$
Thus, we arrive at a contradiction.
Hence (Z, τ_β) is connected.
It is easy to see that every point of Z is cut point of Z.

Theorem: Let X be a connected topological space and x be a cut point of X such that X – {x} = A|B. Then X is open or closed. If {X} is open, A and B are closed and if {X} is closed, A and B are open.

Proof: since X – {x} = A |B, so X – {x} = A ∪ B , $A \neq \emptyset$, $B \neq \emptyset$,
A ∩ $B = \emptyset$, where A, B are both open and closed in X – {x}.
As A is open in X – {x} and X – {x} is a topological subspace of X, so there exist an open subset V of X such that A = V ∩(X – {x})
And so, A = V – {x}. … (1)
Since A is closed in X – {x} and X – {x} is a topological subspace of X, so there exist a closed subset F of X, such that
A = F ∩ ($X -$ {x}) = F – {x}. (2)
By (1) and (2) we get.
A = V – {x} = F – {x} (3)
Let, if possible, V = F
Since A ≠ ∅, so V – {x} ≠ ∅
Also, V ≠ X otherwise by (3), A = X – {x} and so B = ∅.
This gives a contradiction.
Thus, V is open as well as closed set such that V ≠ ∅, V ≠ X.
Hence X is disconnected space, a contradiction.
So, V ≠F.
Thus V – {x} = F – {x}. This implies that either {x} = V – F or {x} = F – V.
If {x} = V – F, then {x} is open in X.
Now by (3) A = F – {x} = F is closed in X. [x∉ F]
If {x} = F – V, then {x} is closed in X
A = V – {x} implies A= V.
This implies that A is open in X.

Remark: Any cut point in a non-empty connected topological space is either open or closed.

Theorem: Let X be a connected topological space and let Y be the subset of all cut points of X. Then the following statements hold.
(a) Every non-empty connected subset of Y, that is not a singleton, contains at least one closed point.
(b)x∈ Y is open, then every limit point of {x} in Y is a closed point.

Proof :(a) Let A be a non-empty connected subset of Y that is not singleton.
Let, if possible, A contains no closed point.
For every x ∈A.
⇒ x∈ Y.
⇒x is a cut point of X.
So either {x} is closed or open.
Since x ∈ A and so by assumption {x} is open.
Also, A is open. [Since A contains no closed point and so every point of A is open].
Therefore A – {x}, {x} form a separation of A.
This implies that A is not connected, a contradiction.
So A contains at least one closed point.
(b) Given that x ∈ Y is open.
Let a be any limit point of {x} in Y and a ∈ Y.
To prove {a} is closed point in Y.
Let if possible {a} is open in Y.
Then a is a neighbourhood of a.
Since a is a limit point of {x}, So every neighbourhood of a in Y, contain at least one point of {x} different from a.
In particular {a} ∩ ({x} – {a}) ≠ ∅, contradiction.
Thus {a} is closed point in Y.

REFERENCE

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