Some Realizable and Nonrealizable Lattice of Convex Edge Sets

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***Abstract*- Let G be a connected directed graph and E(G) be the directed edge set of G. A subset C of E(G) is said to be convex if for any $e_i, e_j \in C$, there is a directed path containing e_i, e_j and the edge set of every $e_i - e_j$ geodesic is contained in C. Let Con(G) be the set of all convex edge sets of G together with empty set partial ordered by set inclusion relation. Then Con(G) forms a lattice if and only if G has an Euler trial. In this paper some realizable and nonrealizable lattice of convex edge sets is discussed.**

***Key words*: Lattices, Chains, Connected digraphs, Convex edge sets**

MSC: 06B99, 05C20, 05C38

1. INTRODUCTION

Motivated by the studies on the lattice of convex sets of a connected graph [11], the set of convex edge sets of connected digraphs together with empty set is considered in [1] and it is found that this set forms a lattice with respect to the partial order set inclusion if and only if digraph contains an Euler trail. In [2] properties of these lattices when the digraph G is directed path and directed cycle are studied. Also irreducibility criteria and conditions under which con(G) becomes lower semimodular is discussed. It is proved that if |E(G)| ≥ 3, Con(G) satisfies lower covering condition if and only if G is a directed cycle C_3. In this paper some realizable and nonrealizable lattice of convex edge sets is discussed.

After introducing some basic concepts, notations and stating some fundamental results, in section 2 we have mentioned <Con(G), ⊆> is a lattice if and only if G contains an Euler trail[1] with an example .

In section 3, the following results are proved. A chain with more than two elements cannot be realized as the lattice of convex edge sets of a connected digraph. A lattice, with $|L| > 2$ containing only one atom cannot be realized as the lattice of convex edge sets of a connected digraph. Let $e_i = \{u, v\}$ be any edge of G with indegree of u = m and outdegree of v = n.

Case(i).If $e_j = \{v, u\}$ is also an edge of G, then $\{e_i\}$ is covered by $m + n - 1$ elements.

Case(ii).If $e_j = \{v, u\}$ is not an edge of G, then $\{e_i\}$ is covered by $m + n$ elements.

In section 4, some realizable and nonrealizable lattice of convex edge sets is obtained.

For terminologies and notations used in this paper we refer to [6] and [7]

2. PRELIMINARIES

Let G be a finite connected digraph. E(G) be the edge set of G. A set C ⊆ E (G) is said to be convex in G if for every two edges e_i, e_j ∈ C, there is a directed path containing e_i, e_j and the edge set of every e_i - e_j geodesic (i.e shortest directed path containing e_i and e_j) is contained in C. In a digraph G, a walk in which no edge is repeated is a (directed) trail. A closed walk in which no edge is repeated is a (directed) circuit. A trail containing all the edges of G is Euler trail and a circuit containing all the edges of G is Euler circuit. An element 'a' of a lattice L is join irreducible if $a = b \vee c$ implies that $a = b$ or $a = c$. 'a' is meet irreducible if $a = b \wedge c$ implies that $a = b$ or $a = c$. An element which is both meet and join irreducible is called doubly irreducible. A lattice L is said to satisfy the lower covering condition if for $a, b \in L$ $a \wedge b \prec b$ implies $a \prec a \vee b$. A lattice L is lower semimodular(LSM) if a$\vee b$ covers both a and b implies that both a and b cover $a \wedge b$.

For a finite connected digraph G, let the set of all convex edge sets in G together with empty set be denoted by Con(G). Define a binary relation ≤ on Con(G) by, for A,B ∈ Con(G), A ≤ B if and only if A ⊆ B. Then clearly ≤ is a partial order on Con(G).

Moreover < Con(G), ⊆> forms a lattice where for A,B ∈Con(G), A ∧ B = A ∩ B and A ∨ B = < A∪B> is the smallest convex edge set containing A∪B. In [1] it is proved that <Con(G), ⊆> is a lattice if and only if G contains an Euler trail. If G has no Euler trail, then <Con(G), ⊆> is a meet semilattice.

For example, the lattice given in Fig 2.2 represents the lattice < Con(G), ⊆> of the connected digraph G given in Fig 2.1.

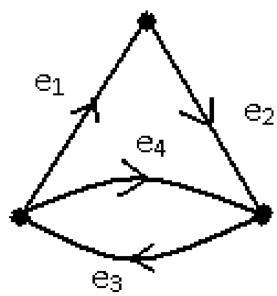


Fig 2.1

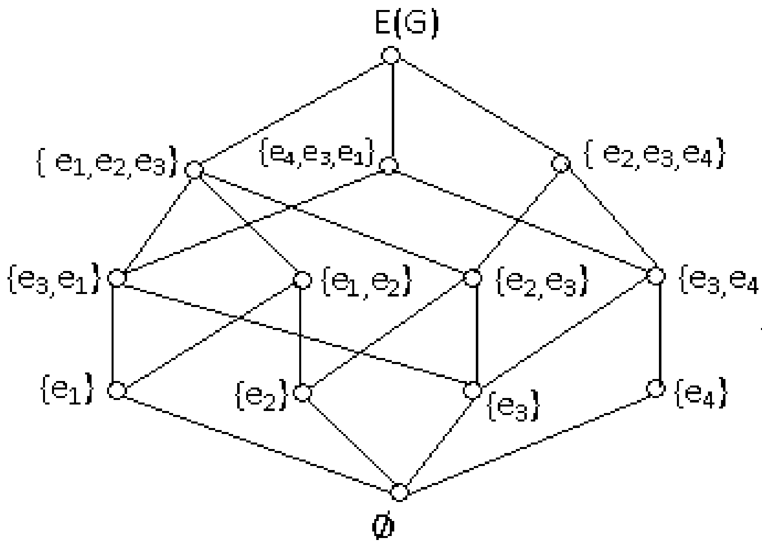


Fig 2.2

Hereafter we consider digraph G containing an Euler trail and use Con(G) to represent the lattice < Con(G), ⊆>

3. ON THE LATTICE CON(G)

Theorem 3.1: A chain with more than two elements cannot be realized as the lattice of convex edge sets of a connected digraph.

Proof: Consider the chain L_n $(n > 2)$, say $\emptyset \prec A_1 \prec A_2 \prec \cdots \prec A_n$. Let this chain be a Con(G) for some connected digraph G. Then $A_1 = \{e_i\}$ for some $e_i \in$ E(G), and $A_2 = <\{e_i\} \cup \{e_j\}>$ for some $e_j \in$ E(G), where $e_j \neq e_i$. Therefore, G must have at least two distinct edges. Then Con(G) will have at least two atoms. Contradiction to the fact that Con(G) is a chain with more than two elements.

Theorem 3.2: A lattice, with $|L| > 2$ containing only one atom cannot be realized as the lattice of convex edge sets of a connected digraph.

Proof: Let L be a lattice containing only one atom say A. Since L can not be a chain there exist at least two elements, say B, C which cover A. If L = Con(G) for some G, then $A = \{e_i\}$ and $B = <\{e_i\} \cup \{e_j\}>$ and $C = <\{e_i\} \cup \{e_k\}>$. Therefore, G must have at least three distinct edges.Then Con(G) must contain at least three atoms, which contradicts the fact that L contains only one atom. Thus a lattice, with $|L| > 2$ containing only one atom cannot be realized as the lattice of convex edge sets of a connected digraph.

Theorem 3.3: Let $e_i = \{u, v\}$ be any edge of G with indegree of u = m and outdegree of v = n.

Case(i).If $e_j = \{v, u\}$ is also an edge of G, then $\{e_i\}$ is covered by $m + n - 1$ elements.

Case(ii).If $e_j = \{v, u\}$ is not an edge of G, then $\{e_i\}$ is covered by $m + n$ elements.

Proof: Let $e_i = \{u, v\}$ be any edge of G with indegree of u = m and outdegree of v = n. Case(i). Let $e_j = \{v, u\}$ is also an edge of G, then G contains distinct edges $e_{1,} e_2 \dots e_{m-1}, e_j$ directed towards the vertex u and $f_{1,} f_2 \dots f_{n-1}, e_j$directed outwards from the vertex v. Therefore $\{e_1, e_i\}, \{e_2, e_i\}, \dots \{e_{m-1}, e_i\}, \{e_j, e_i\}$ and $\{e_i, f_1\}, \{e_i, f_2\}, \dots \{e_i, f_{n-1}\}, \{e_i, e_j\}$ are two elements sets which cover $\{e_i\}$ in Con(G). Since $\{e_j, e_i\} = \{e_i, e_j\}$ there exist $m + n - 1$ distinct elements in Con(G) which cover $\{e_i\}$.

Case(ii).Let $e_j = \{v, u\}$ is not an edge of G, then G contains distinct edges $e_{1,}e_2 \dots e_m$ directed towards the vertex u and $f_{1,}f_2 \dots f_n$ directed outwards from the vertex v. Therefore $\{e_1, e_i\}, \{e_2, e_i\}, \dots \{e_{m-1}, e_i\}, \{e_m, e_i\}$ and $\{e_i, f_1\}, \{e_i, f_2\}, \dots \{e_i, f_{n-1}\}, \{e_i, f_n\}$ are two elements sets which cover $\{e_i\}$ in Con(G). Hence there exist $m+n$ distinct elements in Con(G) which cover $\{e_i\}$.

4. SOME REALIZABLE AND NONREALIZABLE LATTICE OF CONVEX EDGE SETS

Remark 4. 1 : If $|L| = 1$, then it is a trivial lattice, which is realizable as lattice of convex edge sets of a null graph.

Remark 4. 2 : If $|L| = 2$, then it is a two element chain, which is realizable as lattice of convex edge sets of a digraph containing single directed edge.

Remark 4. 3 : If $|L| = 3$, then it is a three element chain, which is not realizable as lattice of convex edge sets. Because a chain with more than two elements cannot be realized as the lattice of convex edge sets of a connected digraph.[Theorem 3.1]

Remark 4. 4 : For $|L| = 4$ there are two lattices up to isomorphism. One is four element chain, which can not be realizable as lattice of convex edge sets.[see Theorem 3.1] . Another is as shown in Figure 4.1a , which is realizable as Con(G), where G is of the form as shown in Figure 4.1.

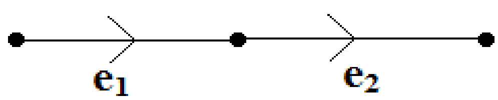


Figure 4.1

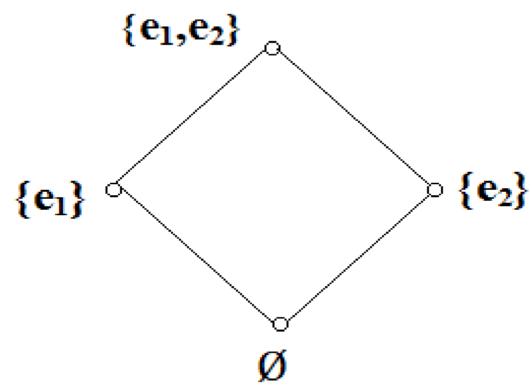


Figure 4.1a

Theorem 4. 5 : For $|L| = 5$, there are five lattices up to isomorphism as shown in Figures 4.2 to 4.6. Among these no lattice is realizable as lattice of convex edge sets.

Figure 4.2

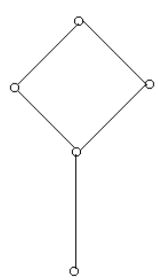
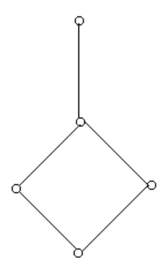


Figure 4.3



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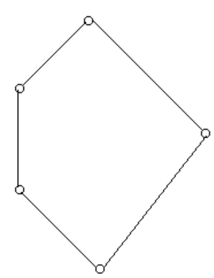


Figure 4.5

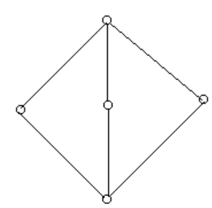
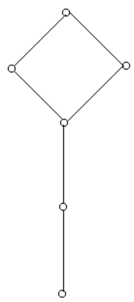


Figure 4.6

Proof: A lattice, with $|L| > 2$ containing only one atom cannot be realized as the lattice of convex edge sets of a connected digraph.[see Theorem 3.2].Therefore Figures 4.2 and 4.3 are not realizable as lattice of convex edge sets. There are only two digraphs with $|E(G)| = 2$. One is cycle of length two and another is directed path of length two, both give the same lattice as shown in Figure 4.1a. Therefore there is only one lattice with two atoms which is realizable as lattice of convex sets as shown in Figure 4.1a. Thus Figures 4.4, 4.5 are not realizable as Con(G). In Con(G) atoms are singleton sets and atoms are covered by two element sets. So each two element set can cover two atoms. Therefore one element (which must be two element set) can not cover three atoms. Thus Figure 4.6 can not be realizable as Con(G).

Theorem 4. 6 : For | L | = 6, there are fifteen lattices up to isomorphism as shown in Figures 4.7 to 4.21. Among these no lattice is realizable as Con(G).


Figure 4.7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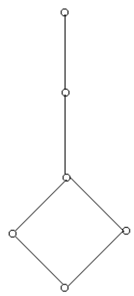
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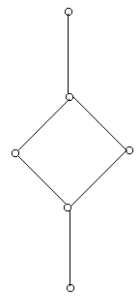
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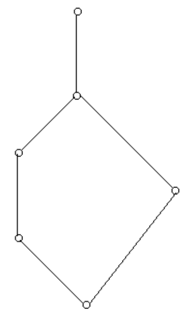
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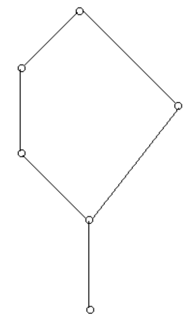
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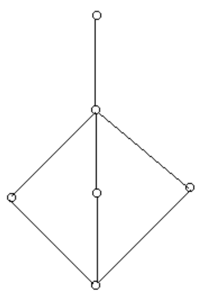
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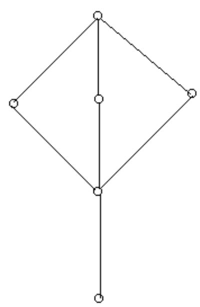
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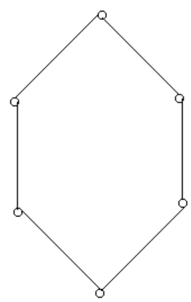
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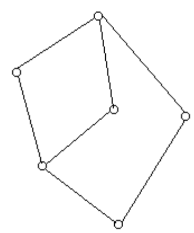
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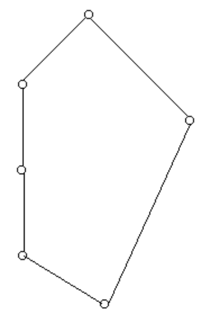
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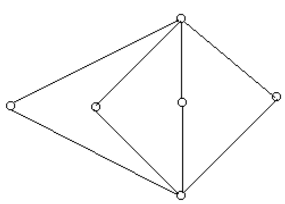
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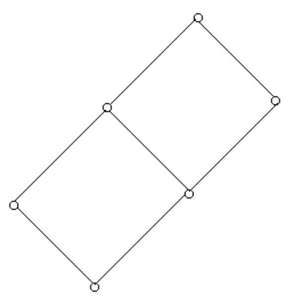
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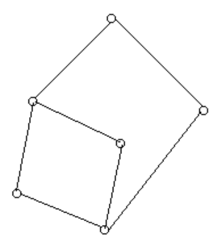


Figure 4.20

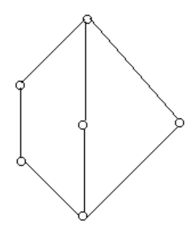
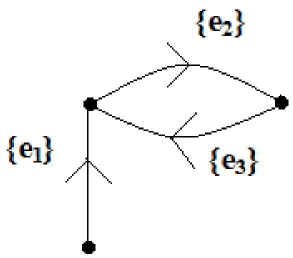


Figure 4.21

Proof: Lattices as shown in Figures 4.7 to 4.19 can not be realizable as lattice of convex edge sets as explained in the previous theorem. Lattices as shown in Figures 4.20, 4.21 contain three atoms. Therefore if they are realizable as Con(G) for some digraph G, then G must have three directed edges. There are four digraphs with three directed edges up to isomorphism. They are as shown in Figures 4.22, 4.23, 4.24 and 4.25. Among these Con(G) for first three lattices will be as shown in Figure 4.22a. Also Con(G) for the digraph as shown in Figure 4.25 is as shown in Figure 4.25a. Therefore these are the only two realizable lattices for a digraph with three edges. Thus Lattices as shown in Figures 4.20 and 4.21 can not be realizable as lattice of convex edge sets.



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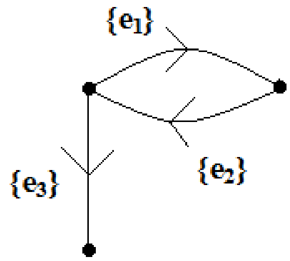

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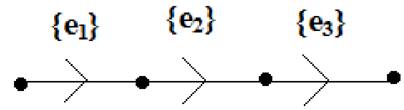

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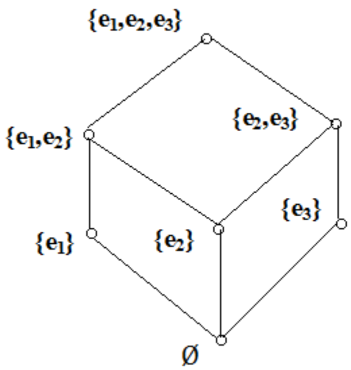

Figure 4.22a

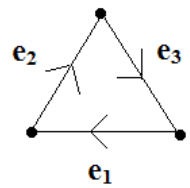

Figure 4.25

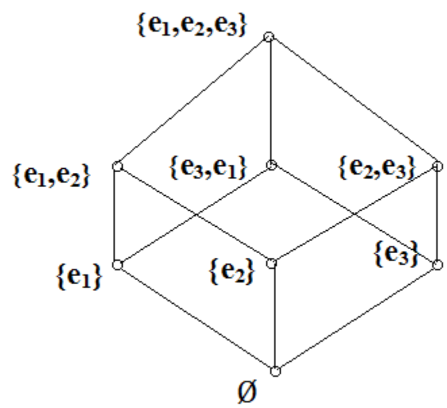

Figure 4.25a

Remark 4. 7 : As | *L* | increases there are more lattices which can be realizable as lattice of convex edge sets of certain graphs.

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