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The Hardy – Ramanujan – Rademacher Expansion For $c\phi_{m,m},(n)$

Mahadevaswamy. B.S.
Department of Mathematics, Maharani's Science College for Women, Mysore, Karnataka, India – 570 005

Abstract **- In this paper we obtain the Hardy – Ramanujan – Rademacher series for $c\phi_{k,h}(n)$ on the lines of L.W.Kolitsch. The existence of such series for $c\phi_{1,k}(n)$ $c\phi_{k,1}(n)$and was asked for by Andrews and later obtained by Kolitsch.**
Finally we extend the results on q-binomial coefficients and q-series representation of Andrews to our function $c\phi_{k,h}(n)$. Andrews has established the two congruences $c\phi_{1-2}(5n + 3) \equiv C\phi_{2-1}(5n + 3) \equiv O \pmod 5$. We whow that the analogous congruence $c\phi_{2,2}(5n + 3) \equiv O \pmod 5$ is false for n = 2. We also study generalised Frobenius partitions with some restriction on its parts.

Index Terms **-Q – binominal co-efficient, Frobenius partitions, analogous congrevence**

INTRODUCTION

Most of the credit in the determination of good asymptotic formulae for p(n) should go to Hardy and Ramanujan [12]. First by elementary reasonings they showed that

$$\log p(n) = \frac{\pi\sqrt{2n}}{\sqrt{3}} + O(\sqrt{n})$$

and then by the use of a Tauberian argument they could show that

$$p(n) = \frac{1}{4n\sqrt{3}} \exp\ [\pi(\frac{2n}{3})^{1/2}]\ (1+O(1)).$$

Finally they showed that the generating function F(x) of p(n) is essentially a modular form. That is, if we change the variable x to $\in^{2\pi it}$ then the denominator of F(x) differs only by a simple factor from

$$\eta(t) = e^{\pi it/12}\prod_{m=1}^{\infty}(1-e^{2\pi imt})$$

By the modular character of F(x), Hardy and Ramanujan were able to apply to F(x) the general theory of Cauchy concerning the determination of the co-efficient in the power series expansion of a known function. In this way sf they found the following expansion of p(n).

(1.1) $$p(n) = \sum_{j=1}^{v} A_j\phi_j + O(n^{-1/4}).$$

where

(1.2.) $$\phi_j = \frac{B_n(u_n - j)\sqrt{j}}{u_n j}.\ e^{u_n/j}$$

$$u_n = \frac{\pi\sqrt{24n} - 1}{6},\quad B_n = \frac{2\sqrt{3}}{24n-1}$$

$$v = O(\sqrt{n})$$

and A's are some constants depending on n and the 24^{th} roots of unity.

At the time (1918) of invention of this formula (1.1) for p(n) it was not known whether the series does or does not coverage. However in 1937, D.H. Lehmer [15] found that the Hardy – Ramanujan expansion (1.1) of p(n) is divergent. Later H.Rademacher [22, 23] showed that if $(u_n - j)\exp(u_n/j)$ is replaced by $(u_n - j)\exp(u_n/j) + (u_n + j)\exp(-u_n/j)$ in (1.2) then we get a convergent series for p(n), that is, an exact formula for p(n). The actual explicit formula for p(n) obtained by Rademacher [23] is the following:

(1.3) $$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)k^{1/2}\left[\frac{d}{dx}\left(\frac{\sinh\left(\frac{\pi}{k}\left[\frac{2}{3}\left(x-\frac{1}{24}\right)\right]^{1/2}\right)}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{x=n},$$

Where $A_k(n) = \sum_{\substack{O<h<k \\ (h,k)=1}} w_{h,k}\exp(-2\pi inh/k)\ with\ w_{h,k}\ a$

Certain 24kth root of unity.

In 1942 P. Erdos [6] proved by entirely elementary considerations that a formula of the type

$$p(n) = An^{-1}\exp[\pi(\frac{2n}{3})^{1/2}]\,(1 + O(1))$$

holds and later in 1951, D.J. Newman [16] showed also by elementary methods that Erdos' constant A was in fact $1/4\sqrt{3}$.

The method of steepest descent employed by G.Szekeres [25, 26] has opened up the possibility of obtaining the infinite series for p(n) without the use of elliptic modular functions.

In [2] George E-Andrews posed the problem of obtaining the Hardy – Ramanujan – Rademacher series for the F-partition functions $\phi_m(n)$ and $c\phi_m(n)$ by a full Farey dissection of the integrals representing then. Recently L.W.Kolitsch [14] obtained the following representations for $\phi_m(n)$ and $c\phi_m(n)$.

(1.4) $$c\phi_m(n) = \frac{1}{\pi\sqrt{2m}}\sum_{k=1}^{\infty} k^{\frac{2-m}{2}} \sum_{\substack{O<s<\beta \\ O\le t\le \mu \\ \frac{t}{m}+2s<\frac{m}{12}}} p_m(s) \Big[\sum_{\substack{O\le h<k \\ (h,k)=1}} \omega_{h,k}^m S(t)\exp[2\pi i(h's-nh)/k] \frac{d}{dx}\Big(\frac{\sinh(\frac{\pi}{k}[(\frac{2m}{3}-\frac{8t}{m}-16s)(x-\frac{m}{24})]^{1/2}}{(x-\frac{m}{25})^{1/2}}\Big)\Big]_{x=n}{}'$$

Where β is the greatest integer < m/24, μ is the greatest integer < $m^2/12$, $p_m(j)$ is the coefficient of q^j in $\prod_{i=1}^{\infty}(1-q^i)^{-m}$, S(t) =

$$\Sigma \sum_{a\in Z_k^{m-1}} \exp[2\pi i(hQ(a_1,\dots,a_{m-1}) + a_1c_1+\dots+a_{m-1}c_{m-1})/k]$$

with the outer sum extending over all solutions of H $(c_1, \dots, c_{m-1})$ – t and H and Q are the quadratic forms defined by

(1.5) $$H(c_1, \dots, c_{m-1}) = c_1^2+\dots+c_{m-1}^2+\sum_{1\le i<j\le m-1}(c_i-c_j)^2$$

(1.6) $$Q(a_1,\dots,a_{m-1}) = \sum_{1\le i<j\le m-1} a_i a_j ,$$

h' satisfies hh' = -1 (mod k) and $\omega_{h,k}$ is a certain 24kth root of unity.

(1.7) $$\phi_m(n) = \frac{1}{\pi\sqrt{2m}}\sum_{k=1}^{\infty} k^{\frac{2-m}{2}} \sum_{\substack{O\le s\le\beta \\ O\le t\le\gamma \\ \frac{t}{m(m+1)^2}+2s<\frac{m}{12}}} p_m(s) \Big[\sum_{\substack{O\le h<k \\ (h,k)=1}} \omega_{h,k}^m T(t)\exp[2\pi i(h's-nh)/k] \frac{d}{dx}\Big(\frac{\sinh(\frac{\pi}{k}[(\frac{2m}{3}-\frac{8t}{m(m+1)}2-16s)(x-\frac{m}{24})]^{1/2}}{(x-\frac{m}{24})^{1/2}}\Big)\Big]_{x=n}{}'$$

Where β, $p_m(j)$, H, Q, h', $w_{h,k}$ are the same as in (1.4) and γ is the greatest integer < $m^2(m+1)^2/12$,

T (t) =

$$\Sigma \sum_{b\in Z^{m-1}k} \exp[2\pi i(hQ(b_1,\dots,b_{m-1})+b_1f_1+\dots+b_{m-1}f_{m-1})/k]$$

=

With the outer sum extending over all solutions of H $(c_1, \dots, c_{m-1})$ = t, $c_i \equiv$ ki (mod m+1) and $f_j = (c_i - ki)/m+1$.

The object of this paper is to obtain the Hardy – Ramanujan – Rademacher series for the generalised Frobenius partition function $c\phi_{m,m'}(n)$ with m colours and m' repetitions. Our discussion is on the lines of Kolitsch and with suitable generalisations of some of his results. By putting m = 1 in our result (Theorem 1) we get Kolitsch's representation (1.7) for $\phi_{m'}(n)$ Substitution m' = 1 in our result gives a representation for $c\phi_{m'}(n)$ which is an alternative to that of Kolitsch.

1. Method of Approach. First we prove a lemma in which we obtain an expansion of the generating function $c\phi_{m,m'}(q)$ of $c\phi_{m,m'}(n)$ in terms of the multidimensional theta functions. While this result contains Theorems 1 and 2 of Andrews [2] as special cases, its proof happens to be on the same lines of Andrews.

   Lemma 1 : For |q| < 1,

(2.1) $$C\phi_{m,m'}(q) = \frac{1}{(q)_\infty^{mm'}}$$

X

$$\sum_{dj_i=-\infty}^{\infty} \zeta^{(m'-1)\sum_{i=1}^{m} d_{1i} + (m'-2)\sum_{i=1}^{m} d_{2i} + \ldots + \sum_{i=1}^{m} d_{m'-1i}} q^{Q(D)},$$

Where

(2.2. $Q(D) = \Sigma\, d_{ji}^{2} + \Sigma\, d_{j'i'}\, d_{j''i''}$,

$\zeta = \exp(2\pi i/m+1)$ and j varies from 1 to m′, i ranges from 1 to m with (j, i) ≠ (m′ , m) and j′, j′ vary from 1 upto m′ with j′ < j′ and i′, i′ range from 1 to m with I′ < I′ .

Proof : From the general Principle of Section we find that $c\phi_{m,m'}(q)$ is the constant term in

(2.3) $CG_{m,m'}(z) =$

$$\prod_{n=O}^{\infty} (1 + zq^{n+1} + \ldots + z^{m'} q^{m'(n+1)})^m$$

$$\text{X } (1+z^{-1}q^{n} + \ldots + z^{-m'} q^{m'n})^m.$$

We can write $C\, g_{m,m'}(z) =$

$$(2.4) = \prod_{n=O}^{\infty} \frac{(1-z^{m'+1} q^{(m'+1)(n+1)})^m\ (1-z^{-m'-1} q^{n(m'+1)})^m}{(1-zq^{n+1})^m\ (1-z^{-1}q^{n})^m}$$

$$= \prod_{j=1}^{m'} \zeta^{j} Z Q_\infty^{m} (\xi^{-jq} z^{-1})_\infty^{m},$$ where $\zeta = \exp(2\pi i/m'+1)$.

$$= \frac{1}{(q)_\infty^{mm}} \Big[\prod_{j=1}^{m'} \sum_{d_j=-\infty}^{\infty} (-1)^{d_j} q^{(d_j+2)} z^{d_j} \zeta^{jd_j} \Big]^m$$

(using Jacobi's triple product identity).

$$= \frac{1}{(q)_\infty^{mm'}} \sum_{\substack{dj_i=-\infty \\ 1\le j\le m' \\ 1\le i\le m}}^{\infty} (-1)^{\sum_{i=1}^{m} d_{ji}}\ q^{\sum_{i=1}^{m} (dji_2^{+1})}\ z^{\sum_{i=1}^{m} d_{ji}}\ \zeta^{\sum_{i=1}^{m} jd_{ji}}$$

The constant term in $CG_{m,m}'(z)$ is obtained by setting $\Sigma\, d_{ji} = O$. That is, $d_{m',m} = -\Sigma\, d_{ji}$ with j = 1, . . . , m′ , i = 1, . . . , m and (j, i) ≠ (m',m).

Consider

(2.5)

$$\sum_{\substack{1\le j\le m' \\ 1\le i\le m}} jd_{ji} = \sum_{\substack{1\le j\le m' \\ 1\le i\le m \\ (j,i)\ne(m',m)}} jd_{ji} + m'd_{m',n}$$

$$= (1-m') \sum_{i=1}^{m} d_{1i} + (2-m') \sum_{i=1}^{m} d_{2i} + \ldots \sum_{i=1}^{m} d_{m'-1i}$$

(substituting for $d_{m',m}$).

Also

(2.6)

$$\sum_{\substack{1\le j\le m' \\ 1\le i\le m}} (^{d}\ ji_{2}^{+1}) = \frac{1}{2} \sum_{\substack{1\le j\le m' \\ 1\le i\le m}} (d_{ji}^{2} + d_{ji})$$

$$= \frac{1}{2} \Big[\sum_{\substack{1\le j\le m' \\ 1\le i\le m \\ (j,i)\ne(m',m)}} (d_{ji}^{2} + d_{ji}) + d_{m'm}^{2} + d_{m'm} \Big]$$

$$= \sum_{\substack{1\le j\le m' \\ 1\le i\le m \\ (j,i)\ne(m',m)}} d_{ji}^{2} + \sum_{\substack{1\le j\le j_1\le m' \\ 1\le i\le i_1\le m \\ (j,i)\ne(m',m)}} d_{ji} d_{j_1 i_1} +$$

(substituting for $d_{m'm}$).

Using (2.5) and (2.6) we find that the constant term in (2.4) is

$$(2.7)\quad \frac{1}{(q)_\infty^{mm'}}$$

X

$$\sum_{\substack{dj_j=-\infty \\ 1\le j\le m' \\ 1\le i\le m \\ (j,i)\ne(m',m)}}^{\infty} \zeta^{(1-m')\sum_{i=1}^{m} d_{1i} + (2-m')\sum_{i=1}^{m} d_{2i} + \ldots + \sum_{i=1}^{m} d_{m'-1i}} q^{Q(D)}$$

Where Q(D) is defined by (2.2). If we change d_{ji} i into $-d_{ji}$ in (2.7), then it becomes the left hand side of (2.1). But the constant term in $CG_{m,m'}(z)$ is $C\phi_{m,m'}(q)$. Thus equating the constant terms we obtain (2.1) and this proves Lemma 1.

Remark : Putting m = 1 in (2.1) we obtain Andrews' representation for $\phi_m'(q)$. The substitution m' = 1 in (2.1) yields Andrews' identity for $C\phi_m(q)$.

To obtain the expansion for $c\phi_{m,m'}(n)$ we use the Hardy – Ramanujan method of Farey fraction dissection of the integral

$$\frac{1}{2\pi i}\int_C \frac{R(q)\,[P(q)]^{mm'}}{q^{n+1}}\,dq,$$

Where C is a circle centered at the origin with radius less than 1,

(2.8) R (q) =

$$\sum_{\substack{dj_j=-\infty \\ 1\le j\le m' \\ 1\le i\le m \\ (j,i)\ne(m',m)}}^{\infty} \zeta^{(m'-1)\sum_{i=1}^{m} d_{1i}+(m'-2)\sum_{i=1}^{m} d_{2i}+\ldots+\sum_{i=1}^{m} d_{m'-1i}} q^{Q(D)}$$

=

Where ζ = exp (2πi/m′+1), P (q) = $\prod_{i=1}^{\infty}$ $(1-q^i)^{-1}$, Q(D) is defined by (2.2). By Cauchy's integral theorem the above integral is equal to $c\phi_{m,m'}(n)$. Our method of approach is similar to that of Kolitsch [14]. However the representation (2.1) is a generalisation.

3. Some Lemmas. In Lemma 2 of this paper we obtain a transformation of our generalised representation (2.1) by using the well-known transformation formula for the multidimensional theta functions. It is a generalisation of Theorem 2.1 of Kolitsch [14]. The proof is similar to Kolitsch's proof of the particular case. A special case of this lemma obtained in Corollary 1 is used to split $c\phi_{m,m'}(n)$ into three convenient sums stated in Lemma 3. These sums are estimated in Lemma 4 and this leads to the proof of our main theorems.

Lemma 2. For all z with Re z > 0.

3.1) R (exp [2πi(iz + h)/k]) =

$$\frac{1}{\sqrt{mm'}}\left(\frac{1}{m'+1}\right)mm'-1\left(\frac{1}{kz}\right)^{\frac{mm'-1}{2}}$$

$$\text{X} \sum_{c\,ji=-\infty}^{\infty} \exp\left(\frac{-\pi H(C)}{mm'(m'+1)^2 kz}\right)$$

$$\sum_{a\,ji=O}^{(m'+1)k-1} \exp\left[2\pi i\left(hQ(A)+\frac{1}{m'+1}\,\Sigma a_{ji}c_{ji}\right)/k\right]$$

Where

(3.2) $H(C) = \Sigma c_{ji}^2 + \Sigma (c_{j'i'} - c_{j''i''})^2$,

Q is as defined in (2.2), the principal branch of $z^{1/2}$ is selected and j′, j′ vary from 1 to m′ with j′ < j′ and i′, i′ range from 1 upto m with I′ < i′ . Here and in what follows j varies from 1 to m′ and i varies from 1 upto m with (j, i) ≠ (m', m).

Proof. : By (2.1) we have

(3.3) R (exp [2πi(iz + h)/k]) =

$$\sum_{dj_j=-\infty}^{\infty} q^{Q(D)} \exp(2\pi i[(m'-1)\sum_{i=1}^{m} d_{li}+\ldots+\sum_{i=1}^{m} d_{m'-li}]/m'+1).$$

Writing

(3.4) d_{ji} = (m' + 1) $kc_{ji} - a_{ji}$,

Where $a_{jj}\varepsilon Z_{(m'+1)}k$ the integers modulo (m′+1)k and $c_{ji\in\in}z$. Substituting (3.4) in (3.3) we obtain

R (exp [2πi(iz + h)/k]) =

$$\sum_{a\varepsilon Z^{mm'-1}_{(m'+1)k}}^{\infty} \exp[2\pi i(\frac{iz+h}{k}Q(A)-\frac{1}{m'+1}\sum_{j=1}^{m'-1}\sum_{i=1}^{m} ja_{m'-ji})]$$

$$\sum_{QZ^{mm'-1}} \exp[-2\pi z(m'+1)^2 kQ(C)]$$

$$\exp(2\pi z(m'+1)[\,\Sigma c_{ji}\,(2a_{ji}+\sum_{(j',i')\ne(j,i)} a_{j'i'})\,])$$

=

$$\sum_{C\varepsilon Z^{mm'-1}_{(m'+1)k}} \exp[2\pi i(\frac{iz+h}{k}Q(A)-\frac{1}{m'+1}\sum_{j=1}^{m'-1}\sum_{i=1}^{m} ja_{m'-1ji})]\theta(x,T)$$

Where θ (x, T) is the multidimensional theta function given by

θ (x, T) =

$$\sum_{c\varepsilon Z^{mm'-1}} \exp\,[2i(c,x) - (c,Tc)]$$

With (,) denoting the inner product of the two column vectors involved, the components of x are x_{ji} = -iz(m′+1) X $2a_{ji}+\sum_{(j',i')\ne(j,i)} a_{j'i'})$ and T is $\pi kz(m'+1)^2$ times the (mm′ – 1) X (mm′ – 1) matrix with 2's on the digonal and 1′s in all other positions. Applying the transformation formula.

θ(x,T) =

$$\pi^{\frac{mm'-1}{2}}|T|^{-1/2}\exp(-\pi^2(x,T^{-1}x))\;\theta(i\pi T^{-1}x,\pi^2T^{-1})$$

We get

(3.5) $R(\exp(2\pi i\,(iz+h)/k]) = \pi\,\frac{mm'-1}{2}\,|\,\Gamma\,|^{-1/2}$

X

$$\sum_{a\in Z^{mm'-1}_{(m'+1)k}} \exp[2\pi i(\frac{iz+h}{k}Q(A)-\frac{1}{m'+1}\sum_{j'=1}^{m'-1}\sum_{i=1}^{m}(m'-j')a_{j'i})]$$

$\exp[-\pi^2(x, T^{-1}x)]\theta(i\pi T^{-1}x, \pi^2T^{-1})$.

We can easily show that |T| = mm′ $[\pi kz(m'+1)^2]^{mm'-1}$ and the matrix T^{-1} is $[\pi kz(m'+1)^2]^{mm'-1}$ times the (mm'-

1) X (mm′-1) matrix with mm′-1 on the disgonal and -1 in all other positions. The components of $\Gamma^{-1}x$ are $-ia_{ji}/\pi$ (m'+1)k,

$-\pi^2 (x, \Gamma^{-1}x) = \frac{2\pi z}{k}(A), (c, i\pi\Gamma^{-1}x) = \frac{1}{m'+1}\Sigma c_{ji}a_{ji},$

$(c, \pi^2\Gamma^{-1}c) = \frac{\pi H(C)}{mm'(m'+1)^2 kz}$ where H is defined by (3.2).

Making these substitutions in (3.3) we obtain

(3.6) R(exp[2πi(iz + h)/k])=

$$\frac{1}{\sqrt{mm'}}\left(\frac{1}{m'+1}\right)^{mm'-1}\left(\frac{1}{kz}\right)^{\frac{mm'-1}{2}}$$

$$\sum_{a\in Z^{mm'-1}_{(m'+1)}} \exp\left[\frac{-2\pi i}{m'+1}\sum_{j'=1}^{m'-1}\sum_{i=1}^{m}(m'-j')a_{j'i} + \frac{2\pi ih}{k}Q(A)\right]$$

$$\sum_{a\in Z^{mm'-1}} \exp\left[\frac{-\pi H(C)}{mm'(m'+1)^2 kz+1} + \frac{2\pi i}{(m'+1)k}\Sigma c_{ji}a_{ji}\right]$$

If we now interchange the order of summation in (3.6) we obtain (3.1) and this proves Lemma 2.

Corollary 1. For all z with Re z > O,

(3.7. R(exp[2πi(iz + h)/k])= $\frac{1}{\sqrt{mm'}}\left(\frac{1}{kz}\right)^{\frac{mm'-1}{2}}$

$$X \sum_{j=O}^{\infty} U(j)\exp\left(\frac{-\pi j}{mm'(m'+1)^2 kz}\right)$$

Where

(3.8. U(j) =

$$\Sigma \sum_{b\varepsilon Z^{mm'-1}} \exp\left[2\pi i(hQ(B) + \Sigma b_{ji} f_{ji})/k\right]$$

With the outer sum extending over all solutions of H(C) = $H(c_1',...,c_{mm'-1}')$ = j where for all i = 1,…,m, $c_{ji} \equiv k(m'-j) \pmod{m'+1}$ for j = 1,…,m′-1 and $c_{m'i} \equiv O \pmod{m'+1}$, while for all i = 1,…,m, $(m'+1)f_{ji} = c_{ji} - k(m' - j)$ for j = 1,…,m′-1 and $(m'+1)f_{m'i} = c_{m'i'}$ $dji \in Z^{mm'-1}_{m'+1}$, b_{ji} vary from O to k-1 and the principal branch of $z^{1/2}$ is selected.

Proof : Replacing a_{ji} by $kd_{ji} + b_{ji}$ where $d_{ji}\in Z_{m'+1}$ and b_{ji} vary from 0, 1,…,k-1 in (3.6), we get

3.9. R(exp[2πi(iz + h)/k])=

$$\frac{1}{\sqrt{mm'}}\left(\frac{1}{\sqrt{m'+1}}\right)^{mm'-1}\left(\frac{1}{kz}\right)^{\frac{mm'-1}{2}}$$

$$\sum_{a\in Z^{mm'-1}} \exp\left[\frac{-\pi H(C)}{mm'(m'+1)^2 kz}\right] \sum_{b_{ji}=0}^{k-1} \exp\left[\frac{-2\pi i}{m'+1}\sum_{j'=1}^{m'-1}\sum_{i=1}^{m}(m'-j')b_{j'i}\right]$$

$$\exp\left[\frac{2\pi i}{k}(hQ(B) + \frac{1}{m'+1}\Sigma c_{ji}b_{j'i})\right] \sum_{d_{ji}\in Z^{mm'-1}_{m'+1}} \exp\left(\frac{2\pi i}{m'+1}\Sigma d_{ji}\alpha_{j'i}\right]$$

Where for all i varying from 1 to m, $\alpha_{ji} = c_{ji} - k(m'-j)$ for j = 1, …,m′-1 and $\alpha_{m'1} = c_{m'1}$.

Since $\sum_{d_{ji}\varepsilon Z^{mm'-1}_{m'+1}} \exp\left(\frac{2\pi i}{m'+1}\Sigma d_{ji}\alpha_{j'i}\right)$

$$= \Pi \sum_{d_{ji}=0}^{m'} \exp\left(\frac{2\pi i}{m'+1} d_{ji}\alpha_{j'i}\right)$$

= $(m'+1)^{mm'-1}$ if m′+1 divides α_{ji} and 0 otherwise, setting $c_{ji} = (m'+1)f_{ji} + k(m'-j)$ for j = 1,…,m′-1 and $c_{m'i} = (m'+1)f_{m'i}$ for all i = 1,…,m in (3.9) it reduces to (3.7) and this establishes corollary 1

To obtain the Farey fraction dissection of

$$c\phi_{m,m'}(n) = \frac{1}{2\pi i}\int_C \frac{R(q)[p(q)]^{mm'}}{q^{n+1}} dq$$

we first set q = ρ exp (2πiϕ), 0 ≤ϕ≤ 1, |ρ| < 1 and then set ρ = exp $(-2\pi N^{-2})$, We then get

(3.10) $c\phi_{m,m'}(n) = \int_o^1 R(\exp[2\pi i(iz+h)/k])$

$[p(\exp[2\pi i(iN^{-2} + \phi)])]^{mm'} \exp(2\pi nN^{-2}) - 2\pi in\phi)\, d\phi.$

Using the notation of [1], we define $\theta'_{h,k}$ and $\theta''_{h,k}$ for (h, k) = 1 as follows :

$\theta'_{h,k} = \frac{1}{N+1}$ h = 0, k = 1

$\theta'_{h,k} = \frac{h}{k} - \frac{h'+h}{k'+k}$ O < h < k

$\theta'_{h,k} = -\frac{h''+h}{k''+k} - \frac{h}{k}$

O ≤ h < k

Where $\frac{h'}{k'}$, $\frac{h}{k}$ and $\frac{h''}{k''}$ are three successive terms in the Farey fraction sequence of order N. Let $\zeta_{h,k}$ denote the interval $-\theta_{h,k}^{\backslash} \le \phi \le \theta'_{h,k'}$.

Lemma 3. $c\phi_{m,m'}(n) = [\frac{1}{\sqrt{mm'}} \sum_{\substack{k=1 \\ O\le h<k \\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} (\sum_{j=o}^{\partial} U(j) \exp(\frac{-\pi j}{mm'(m'+1)^2 kz}))$$

exp

$$[\frac{mm'\pi}{12kz} + \frac{2\pi z}{k}(n - \frac{mm'}{24})] (\sum_{j=O}^{\beta'} p_{mm'}(j) \exp[2\pi i(h'+iz^{-1})\, j/k])\, d\phi]$$

$$+ [\frac{1}{\sqrt{mm'}} \sum_{\substack{k=1 \\ O\le h<k \\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} (\sum_{j=\partial+1}^{\partial} U(j) \exp(\frac{-\pi j}{mm'(m'+1)^2 kz}))$$

exp

$$[\frac{mm'\pi}{12kz} + \frac{2\pi z}{k}(n - \frac{mm'}{24})] (\sum_{j=O}^{\beta'} p_{mm'}(j) \exp[2\pi i(h'+iz^{-1})\, j/k])\, d\phi]$$

$$+ [\frac{1}{\sqrt{mm'}} \sum_{\substack{k=1 \\ O\le h<k \\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} (\sum_{j=\partial+1}^{\partial} U(j) \exp(\frac{-\pi j}{mm'(m'+1)^2 kz}))$$

exp

$$[\frac{mm'\pi}{12kz} + \frac{2\pi z}{k}(n - \frac{mm'}{24})] (\sum_{j=\beta'+1}^{\infty} p_{mm'}(j) \exp[2\pi i(h'+iz^{-1})\, j/k])\, d\phi]$$

Where ∂ is the greatest integer > $m^2m^2(m'+1)^2/12$, β' is the greatest integer < $mm'/24$, $p_{mm'}(j)$ is the coefficient of q^j in $[p(q)]^{mm'}$ where $P(q) = \prod_{i=1}^{\infty}(1-q^i)^{-1}$ and $\omega_{h,k}$ is a certain 24kth root of unity.

Proof : Dividing the interval of integration at the mediants of the Farey fraction sequence of order N and replacing ϕ by $\phi + \frac{h}{k}$ and z by $k(N^{-2} - i\phi)$ in (3.10) we obtain

(3.11) . $c\phi_{m,m'}(n) = \sum_{\substack{k=1 \\ O\le h<k \\ (h,k)=1}}^{N} \exp(-2\pi inh/k)$

$$\int_{\zeta_{h,k}} R(\exp[2\pi i(iz+h)/k])\,[p(\exp[2\pi i(iz+h)])]^{mm'} \exp(2\pi nz/k)\, d\phi.$$

Using Lemma 2 and the following transformation formula for all z with Re z > 0.

$P(\exp[2\pi I(iz+h)/k]) = \omega_{h,k}\, z^{1/2} \exp[\pi(z^{-1} - z)/12k]$

$X\ P(\exp[2\pi i(h' + iz^{-1})/k])$

Where (h,k) = 1, h′ satisfies $hh' \equiv -1 \pmod k$, $\omega_{h,k}$ is a 24kth root of unity and the principal branch of $z^{1/2}$ is selected, we find that (3.11) becomes

$$c\phi_{m,m'}(n) = \frac{1}{\sqrt{mm'}} \sum_{\substack{k=1 \\ O\le h<k \\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} (\sum_{j=0}^{\infty} U(j) \exp(\frac{-\pi j}{mm'(m'+1)^2 kz}))$$

exp

$$[\frac{mm'\pi}{12kz} + \frac{2\pi z}{k}(n - \frac{mm'}{24})]\,[p(\exp[2\pi i(h'+iz^{-1})/k])^{mm'}\, d\phi]$$

The result stated in Lemma 3 is obtained by splitting the above expression of $c\phi_{m,m'}(n)$ into three sums as indicated.

We now find the estimates of the three sums say Σ_1, Σ_2 and Σ_3 (respectively) stated in Lemma 3. As in [14] we show that Σ_1 contributes the principal estimate for $c\phi_{m,m'}(n)$ and the contributions from Σ_2 and Σ_3 are negligible. For this we show that $|\Sigma_2|$ and $|\Sigma_3|$ approach zero as N tends to infinity for n fixed. To establish this we first find the bounds for the integrands in Σ_2 and Σ_3.

Considering the integrand in Σ_2 we have

$$|\ z^{1/2} k^{(1-mm')/2} \sum_{j=\partial+1}^{\infty} (U(j) \exp \frac{-\pi j}{mm'(m'+1)^2 kz}$$

exp

$$[\frac{mm'\pi}{12kz} + \frac{2\pi z}{k}(n - \frac{mm'}{24})] (\sum_{j=o}^{\infty} p_{mm'}(j) \exp[2\pi i(h'+iz^{-1})\, j/k])|$$

$$<|z^{1/2}|\, k^{(1-mm')/2} \sum_{j=\partial+1}^{\infty} |U(j)| \exp[\frac{-\pi}{k}(\frac{j}{mm'(m'+1)^2} - \frac{mm'}{12}) \operatorname{Re}\frac{1}{z}])$$

$$\exp[2\pi(n-\frac{mm'}{24})\operatorname{Re}\frac{z}{k}](\sum_{j=0}^{\infty} p_{mm'}(j)\exp[\frac{-2\pi i}{k}\operatorname{Re}\frac{1}{z}])$$

$$<2^{1/4} N^{-1/2} k^{(1-mm')/2} \sum_{j=\partial+1}^{\infty} r(j)(mm'k)^{\frac{mm'-1}{2}}$$

exp

$$[\frac{-\pi}{2}\frac{j}{mm'(m'+1)^2}-\frac{mm'}{12}+2\pi(n-\frac{mm'}{24})N^{-2}](\sum_{j=o}^{\infty} p_{mm'}(j)\exp(-\pi j)$$

(where r(j) is the number of solutions of $H(c_1',...,c_{mm'-1}')=j)$ since $|z^{1/2}|< 2^{1/4} N^{-1/2}$ and $\frac{1}{k}$

$\operatorname{Re}\frac{1}{z}>\frac{1}{2}$ for $\phi\varepsilon^{\zeta_{h,k}}$ and $|U(j)| < r(j)\ (mm'k)^{(mm'-1)/2}$.

(3.12)

$$|\sum_{a\in Z_k^{m-1}} \exp[\frac{2\pi i}{k}(hQ(a_1,...,a_{m-1})+\sum_{j=1}^{m-1} c_j a_j)]|<(mk)^{\frac{m-1}{2}}$$

Since the two sums in the above estimate of the integrand of Σ_2 are convergent it is easy to see that the integrand is bounded in absolute value by $c_1N^{-1/2}$ exp $(2\pi nN^{-2})$, where c_1 is a constant independent of N.

Similarly for the integrand in Σ_3, we have

$$| z^{1/2} k^{(1-mm')/2} \sum_{j=0}^{\partial} U(j)\ \exp\ \frac{-\pi j}{mm'(m'+1)^2 kz}))$$

exp

$$[\frac{mm'\pi}{12kz}+\frac{2\pi z}{k}(n-\frac{mm'}{24})](\sum_{j=\beta'+1}^{\infty} p_{mm'}(j)\exp[2\pi i(h'+iz^{-1})\ j/k])\ |$$

$$<| z^{1/2} k^{(1-mm')/2} \sum_{j=0}^{\partial}|U(j)|\exp[(\frac{-\pi j}{mm'(m'+1)^2 k}\ \operatorname{Re}\frac{1}{z}])\ |$$

$$\exp[2\pi(n-\frac{mm'}{24})\operatorname{Re}\frac{z}{k}](\sum_{j=\beta'+1}^{\infty} p_{mm'}(j)\exp[\frac{-2\pi i}{k}(j-\frac{mm'}{24})\operatorname{Re}\frac{1}{z}])$$

$$<2^{1/4} N^{-1/2} k^{(1-mm')/2}\ (\sum_{j=0}^{\partial} r(j)(mm'k)^{\frac{mm'-1}{2}}\ \exp[\frac{-\pi j}{2mm'(m'+1)^2}])$$

$$\exp[2\pi N^{-2}(n-\frac{mm'}{24})](\sum_{j=\beta'+1}^{\infty} p_{mm'}(j)\exp[-\pi(j-\frac{mm'}{24})])$$

It is easy to show that this is bounded by $c_2N^{-1/2}$ exp $(2\pi nN^{-2})$. Where c_2 is a constant independent of N.

Thus Σ_2 and Σ_3 are bounded in absolute value by

$$cN^{-1/2}\ \exp(2\pi nN^{-2}) \sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \int_{\zeta_{h,k}} d\phi = cN^{-1/2} \exp(2\pi nN^{-2}) \int_O^1 d\phi$$

$$= cN^{-1/2} \exp(2\pi nN^{-2})$$

Where c is a constant independent of N. Clearly, for n fixed this approaches zero as N tends to infinity.

We now consider the first sum Σ_1 in Lemma 3.

$$\Sigma_1 = \frac{1}{\sqrt{mm'}} \sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} \sum_{j=0}^{\partial} U(j)\exp\ (\frac{-\pi j}{mm'(m'+1)^2 kz})$$

exp

$$[\frac{mm'\pi}{12kz}+\frac{2\pi z}{k}(n-\frac{mm'}{24})](\sum_{j=0}^{\beta'} p_{mm'}(j)\exp[2\pi i(h'+iz^{-1})\ j/k])\ d\phi$$

$$= \frac{1}{\sqrt{mm'}} \sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{1/2} k^{(1-mm')/2} \exp[\frac{mm'\pi}{12kz}+\frac{2\pi z}{k}(n-\frac{mm'}{24})]$$

$$\sum_{\substack{O\le t\le \partial\\ O\le s\le \beta'}} U(t)\ p_{mm'}(s)\exp[\frac{2\pi ih's}{k}-\frac{\pi}{kz}(\frac{t}{mm'(m'+1)^2}+2s)]\ d\phi.$$

We separate this into the sums as follows :

$$[\frac{1}{\sqrt{mm'}} \sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} (\frac{z}{k})^{\frac{1}{2}} k^{\frac{2-mm'}{2}} \exp[\frac{2\pi z}{k}(n-\frac{mm'}{24})]$$

$$\sum_{\substack{O\le t\le\partial\\ O\le s\le\beta'\\ \frac{t}{mm'(m'+1)^2}+2s\ge\frac{mm'}{12}}} U(t)$$

$$P_{mm'}(s)$$

$$\exp[\frac{2\pi ih's}{k}+\frac{\pi}{kz}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s)]\ d\phi.$$

$$+[\frac{1}{\sqrt{mm'}} \sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'} \exp(-2\pi inh/k)$$

$$\int_{\zeta_{h,k}} z^{\frac{1}{2}} k^{\frac{1-mm'}{2}} \exp[\frac{2\pi z}{k}(n-\frac{mm'}{24})]$$

$$\sum_{\substack{O\le t\le \partial \\ O\le s\le \beta' \\ \frac{t}{m(m+1)^2}+2s\ge \frac{mm'}{12}}} U(t)$$

$P_{mm'}(s)$

$$\exp[\frac{2\pi i h's}{k}+\frac{\pi}{kz}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}+2s)]\ d\phi.$$

$= \Sigma_{1a} + \Sigma_{1b}$, (say),

Now we show that Σ_{1b} is bounded in absolute value by a constant times $N^{-1/2}\exp(2\pi nN^{-2})$. For this we consider the integrand in Σ_{1b}.

$$\Big| z^{\frac{1}{2}} k^{\frac{1-mm'}{2}} \exp[\frac{2\pi z}{k}(n-\frac{mm'}{24})]$$

$$\sum_{\substack{O\le t\le \partial \\ O\le s\le \beta' \\ \frac{t}{mm'(m'+1)^2}+2s\ge \frac{mm'}{12}}} U(t)$$

$P_{mm'}(s)$

$$\exp[\frac{2\pi i h's}{k}+\frac{\pi}{kz}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s)]\Big|$$

$$<|z^{\frac{1}{2}} k^{\frac{1-mm'}{2}} \exp[2\pi(n-\frac{mm'}{24})\ \mathrm{Re}\,\frac{z}{k}]$$

$$\sum_{\substack{O\le t\le \partial \\ O\le s\le \beta' \\ \frac{t}{mm'(m'+1)^2}+2s\ge \frac{mm'}{12}}} |U(t)|\, p_{mm'}(s)$$

$$\exp[\frac{\pi}{k}\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}+2s)\ \mathrm{Re}\,\frac{1}{z}]\,|$$

$$< 2^{\frac{1}{4}} N^{\frac{-1}{2}} k^{\frac{1-mm'}{2}} \exp[2\pi(n-\frac{mm'}{24})N^{-2}]$$

$$\sum_{\substack{O\le t\le \partial \\ O\le s\le \beta' \\ \frac{t}{mm'(m'+1)^2}+2s\ge \frac{mm'}{12}}} \gamma(t)$$

$(mm'k)^{(mm'-1)/2}$ $P_{mm'}(s)$

$$\exp[\frac{\pi}{2}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s)].$$

It is easy to show that this is bounded by $c_3N^{-1/2}\exp(2\pi nN^{-2})$, where c_3 is a constant independent of N. As before, $|\Sigma_{1b}|$ $c_3(mm')^{-1/2} N^{-1/2}\exp(2\pi nN^{-2})$ which tends to zero as N tends to infinity for fixed n.

We now consider the integral appearing in Σ_{1a}.Replacing z by kw this integral becomes

$$\int_{\zeta_{h,k}} w^{1/2} \exp[2\pi w(n-\frac{mm'}{24})+\frac{\pi}{k^2w}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)}2-2s))\,d\phi$$

Since $z = k(N^{-2} - i\phi)$,we have $w = N^{-2} - i\phi$. Also since $\zeta_{h,k}$ represents the interval $-\theta'_{h,k}\le\phi\,\theta''_{h,k}$, the above integral can be written as

$$\int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} w^{\frac{1}{2}} \exp[2\pi w(n-\frac{mm'}{24})+\frac{\pi}{k^2w}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)}2-2s)]\,(\frac{-1}{i})\,d\omega$$

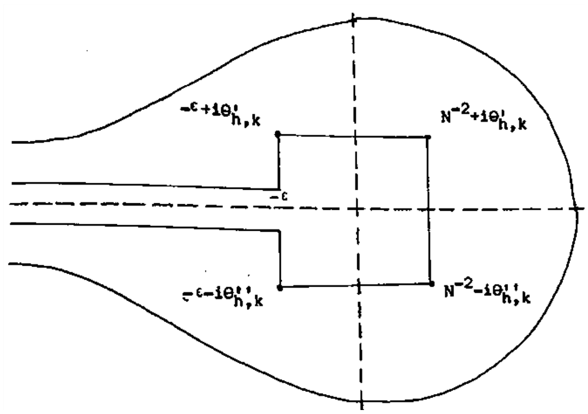
=

$$\frac{1}{i}\Big(\int_{-\infty}^{(0+)}-\int_{-\infty}^{-\varepsilon}-\int_{-\varepsilon}^{-\varepsilon-i\theta'_{h,k}}-\int_{-\varepsilon-i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}}-\int_{N^{-2}+i\theta'_{h,k}}^{-\varepsilon+i\theta'_{h,k}}$$

$$-\int_{-\varepsilon+i\theta^{i}_{h,k}}^{-\varepsilon}-\int_{-\varepsilon}^{-\infty}\Big)w^{\frac{1}{2}}\exp[2\pi w(n-\frac{mm'}{24})+\frac{\pi}{k^2w}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)}2-2s)]\,d\omega$$

(where we assume $c < N^{-2}$ and $\int_{-\infty}^{(O+)}$ represents the loop integral along the contour.

$= \frac{1}{i}$ $(L_k - I_1 - I_2 - I_3 - I_4 - I_5 - I_6)$, say.



w- plane cut along the negative real axis

We now show that the sums associated with I_2, I_3, I_4 and I_5 are negligible. For this, we first find the bounds for them.

Replacing w by - $\in$ + iv, we get $|I_2|$ =

$$| \int_{O}^{-\theta''_{h,k}} (-\varepsilon+iv)^{\frac{1}{2}} \exp[2\pi(n-\frac{mm'}{24})(-\varepsilon+iv)+\frac{\pi\alpha}{k^2(-\varepsilon+iv)}]dv |$$

$$(where\ \alpha=\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s).$$

$$< \int_{O}^{-\theta''_{h,k}} (\varepsilon^2+v^2)^{\frac{1}{4}} \exp[-2\pi(n-\frac{mm'}{24})+\frac{\pi\alpha}{k^2}\mathrm{Re}\frac{1}{-\varepsilon+iv})]|dv|$$

$$<\varsigma(\varepsilon^2+\theta''_{h,k})^{1/4}\exp[-2\pi(n-\frac{mm'}{24})]\ \theta''_{h,k}\ ,$$

Since $\mathrm{Re}\ \frac{1}{-\varepsilon+iv} = \frac{1}{\varepsilon^2+v^2} < O.$

<

$$(\varepsilon^2+k^{-2}N^{-2})^{1/4}(kN)^{-1}\exp[-2\pi(n-\frac{mm'}{24})],\ \sin ce\ \theta''_{h,k} < (kN)^{-1}$$

This approaches $k^{-3/2}$ $N^{-3/2}$ as ε tends to zero.

We can similarly show that I_5 is also bounded by $k^{-3/2}$ $N^{-3/2}$ (as ε tends to zero) and so the sums of I_2 and I_5 are bounded in absolute value by a constant (independent of N) times $N^{-1/2}$. On exactly similar lines we can show that I_3 and I_4 are bounded in absolute value by $2^{1/4}$ $k^{-1/2}$ $N^{-5/2}$ exp $(2\pi nN^{-2} + mm'\pi/3)$ (as ε approaches zero) and the sums associated with these are bounded in absolute value by N-1/2 exp $(2\pi nN^{-2})$. All these bounds tend to zero (for n fixed) as N tends to infinity. Thus we have proved the following lemma.

Lemma 4. $c\phi_{m,m'}$ (n) = $\frac{1}{\sqrt{mm'}}$

$$\sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'}\, k^{\frac{2-mm'}{2}} \exp(-2\pi inh/k)$$

$$\sum_{\substack{O\le t\le\partial\\ O\le s\le\beta'\\ \frac{t}{mm'(m'+1)^2}+2s<\frac{mm'}{12}}} U(t)\,p_{mm'}(s)\exp(2\pi ih's/k)$$

$[\frac{1}{i}(L_k - I_1 - I_6)]$ + $O(N^{-1/2})$ + $O(N^{-1/2}\exp(2\pi nN^{-2}))$.

4. The Main Theorem.

Theorem 1.

$$c\phi_{m,m'}(n)=\frac{1}{\pi\sqrt{2mm'}}\sum_{k=1}^{\infty} k^{\frac{2-mm'}{2}}$$

$$\sum_{\substack{O\le t\le\partial\\ O\le s\le\beta'\\ \frac{t}{mm'(m'+1)^2}+2s<\frac{mm'}{12}}} p_{mm'}(s)[\sum_{\substack{O\le h<k\\ (h,k(=1)}} u(t)\omega_{h,k}^{mm'}\exp[2\pi i(h's-nh)/k]$$

$$\frac{d}{dx}(\frac{\sinh(\frac{\pi}{k}[(\frac{2mm'}{3}-\frac{8t}{mm'(m'+1)}2-16s)(x-\frac{mm'}{24})]^{1/2}}{(x-\frac{mm'}{24})^{1/2}})]_{x=n}$$

Proof: If we let $\in$ tend to zero then $L_k - I_1 - I_6$ can be replaced by the integral.

$$\int_K w^{1/2}\exp[2\pi w(n-\frac{mm'}{24})\frac{\pi}{k^2 w}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s)]\ dw,$$

Where K is the circle $|w-\frac{1}{2}|=.\frac{1}{2}$ travelled counterclockwise from the origin. The substitution w = 1/v changes the above integral into

$$\int_{1-i\infty}^{1+i\infty} v^{-5/2}\exp[\frac{2\pi}{v}(n-\frac{mm'}{24})+\frac{\pi v}{k^2}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)}2-2s)]\ dv,$$

Since $\frac{\pi}{k}(\frac{mm'}{12}-\frac{t}{mm'(m'+1)^2}-2s)$ is positive this integral can be expressed in terms of Bessel functions as follows :

$$\frac{i}{\pi\sqrt{2}}[\frac{d}{dx}(\frac{\sinh(\frac{\pi}{k}[(\frac{2mm'}{3}-\frac{8t}{mm'(m'+1)^2}-16s)(x-\frac{mm'}{24})]^{1/2})}{(x-\frac{mm'}{24})^{1/2}})]_{x=n}$$

Substituting this in the formula for $c\phi_{m,m'}(n)$ obtained in Lemma 3, we get

$c\phi_{m,m'}$ (n) = $\frac{1}{\sqrt{mm'}}$

$$\sum_{\substack{k=1\\ O\le h<k\\ (h,k)=1}}^{N} \omega_{h,k}^{mm'}\, k^{(2-mm')1/2}\exp(-2\pi inh/k)$$

$$\sum_{\substack{O \le t \le \partial \\ O \le s \le \beta' \\ \frac{t}{mm'(m'+1)^2} + 2s < \frac{mm'}{12}}} U(t)\, p_{mm'}(s)\, \exp(2\pi i h' s / k)$$

$$\frac{i}{\pi\sqrt{2}} \left[\frac{d}{dx} \left(\frac{\sinh\left(\frac{\pi}{k} \left[\left(\frac{2mm'}{3} - \frac{8t}{mm'(m'+1)} 2 - 16s \right) \left(x - \frac{mm'}{24} \right) \right]^{1/2} \right)}{\left(x - \frac{mm'}{24} \right)^{1/2}} \right) \right]_{x=n}$$

$+ O(N^{-1/2}) + O(N^{-1/2} \exp(2\pi n N^{-2}))$.

If we now let N approach infinity, we obtain the desired identity for $c\phi_{m,m'}(n)$.

REFERENCE

[1] G.E. Andrews, The Theory of Partitions, Encyclopedia of Math. And Its Applications, Vol. 2, Addison – Wesley, Reading, 1976.

[2] G.E. Andrews, Generalised Frobenius Partitions, Memoirs of the Amer.Math.Soc. Vol. 49, No. 301 (1984), 1-44.

[3] G.E. Andrews, Hecke Modular Forms and the Kac – Peterson Identities, Trans. Amer. Math. Soc. Vol. 283, No.2 (1984), 451-458.

[4] G.E. Andrews, q-series, Their Development and Application in Analysis, Number Theory, Combinatories, Physics and Computer Algebra, Amer. Math.Soc. 1986.

[5] R.Bellman, A Brief Introduction to Theta Functions, Holt, Rinehart and Winston, New York, 1961.

[6] P. Erdos and J. Lehner, The Distribution of the Number of Summands in the Partitions of a Positive Integer, Duke Math.J.8 (1941), 335-345.

[7] L. Euler, Introduction on Analysis in Infinitorum, vol. I, Marcum-Michaelem Bousquet, Lausannae, 1748.

[8] G. Frobenius, Uber die Charaktere der Symmetrischen Gruppe, Sitzber, Preuss.Akad.Berlin, (1900), 516-534.

[9] C.F. Gauss, Werke Vol.2, Konigliche Gesselschaft der Wissenschaften, Gottingen, 1863.

[10] B. Gordon, Two New Representations of the Partition Function, Proc.Amer.Math.Soc.13 (1962), 869-873.

[11] B.Gordon and L.Houten, Notes on Plane Partitions II, J.Comb.Th.4 (1968), 81-99.

[12] G.H.Hardy and S.Ramaujan, Asymptotic Formulae in Combinatory Analysis, Proc.London Math.Soc.(2), 17 (1918), 75-115.

[13] C.G.J.Jacobi, Fundamenta Nova Theoriae Funktonum Ellipticarum, Regiomenti Fratrum Borntrager, 1829 (reprinted in Gesammelte Werke, Vol. 1, 49-239, Reimer, Berlin, 1881).

[14] L.W. Kolitsch, Some Analytic and Arithmetic Properties of Generalised Frobenium Partitions, Ph.D Thesis, Pennsylvania State University, May 1985.

[15] D.H. Lehmer, On the Hardy-Ramanujan Series for the Partition Function, J. London Math.Soc.12 (197), 171-176.

[16] D.J.Newman, The Evaluation of the Constant in the Formula for the Number of Partitions of n, Amer.Jour.Math.73 (1951), 599-601.

[17] Padmavathamma, On Andrews' Generalised Frobenius Partitions, Fibonacci Quart. (to appear).

[18] Padmavathamma, The Hardy-Ramanujan-Rademacher Expansion for $c\phi_{m,m'}(n)$, Journal of the Ramanujan Mathematical Society, II Issue (1987), (to appear).

[19] Padmavathamma, Some Results Concerning Generalised Frobenius Partitions, Bull.Calcutta Math.Soc. (to appear).

[20] Padmavathamma, Restricted Generalised Frobenius Partitions, Indian J.Pure Appl.Math. (to appear).

[21] H.Rademacher, Topics in Analytic Number Theory, Springer-Verlag, Berlin and Heidelberg, 1973.

[22] H.Rademacher, On the Partition Function p(n), Proc. London Math.Soc.43 (1937), 241-254.

[23] H.Rademacher, On the Expansion of the Partition Function in a Series, Ann.Maths.44 (1943), 416-422.

[24] S.Ramanujan, Collected Papers, Chelsea, New York, 1962.

[25] G.Szekeres, An Asymptotic Formula in the Theory of Partitions, Quart.J.Math.Oxford Ser.(2) 2 (1951), 85-108.

[26] G.Szekeres, Some Asymptotic Formulae in the Theory of Parittions, II, Quart.J.Math.Oxford Ser.(2) 4. (1953), 96-111.