Binet-Curve for the various Left $\boldsymbol{k}$-Gaussian Fibonacci sequences

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Abstract—**In this article, we trace Binet-curves for some specific Left k-Gaussian Fibonacci numbers $GF_n^{L(a,b)}$. We extend their Binet-type formula to arbitrary real variables in complex plane and use them to analyze and calculate the area under these curves.**

Index Terms— **Curve tracing, Gaussian Fibonacci sequence, Gaussian Lucas sequence, Gaussian Pell sequence, Gaussian Pell-Lucas sequence.**

I. INTRODUCTION

Many papers about a variation of generalizations of Fibonacci sequence have appeared in recent years. (See: [1,3,4,6]). There are fundamentally two ways in which the Fibonacci sequence may be generalized; either by preserving the recurrence relation but varying the first two terms of the sequence from 0, 1 to arbitrary integers a, b or by preserving the first two terms of the sequence but altering the recurrence relation. Both the methods can be united, but a change in the recurrence relation lead to greater difficulty in the properties of the resulting sequence.

In this paper, we consider the generalized Fibonacci sequences which uses complex numbers, namely: the Left k-*Gaussian Fibonacci sequence* $\left\{GF_n^{L(a,b)}\right\}$ defined by the recurrence relation

$$GF_n^{L(a,b)} = kGF_{n-1}^{L(a,b)} + GF_{n-2}^{L(a,b)}, \text{ for } n \geq 2 \tag{1}$$

with $GF_0^{L(a,b)} = a + i(b - ka)$ and $GF_1^{L(a,b)} = b + ia$, where k, a and b are any positive integers. First few terms of this sequence are $a + i(b - ka), b + ia, (kb + a) + ib, (k^2b + ka + b) + i(kb + a), (k^3b + k^2a + 2kb + ka + a) + i(k^2b + b)$. This sequence can be expressed as a function of the roots $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$ of the characteristic equation $x^2 - kx - 1 = 0$ associated with the recurrence relation for this sequence. Here we note that for these values we have $\alpha > \beta$, $\alpha - \beta = \sqrt{k^2+4}$ and $\alpha\beta = -1$. The equivalent extended Binet type formula for this sequence is given by $GF_n^{L(a,b)} = \frac{h\alpha^n - l\beta^n}{\alpha-\beta}$, where $h = \alpha a + (b - ka) + i\{a - \beta(b - ka)\}$ and $l = \beta a + (b - ka) + i\{a - \alpha(b - ka)\}$. When we choose different values of k, a, b, we obtain sequences like Gaussian Fibonacci sequence, Gaussian Lucas sequence, Gaussian Pell sequence, Gaussian Pell-Lucas sequence, For further details about this sequence, (See: [7]).

The Gaussian Fibonacci sequence $\{GF_n\}$ is defined by the recurrence relation $GF_n = GF_{n-1} + GF_{n-2}$, for all $n \geq 2$ with initial conditions $GF_0 = i$ and $GF_1 = 1$. The first few terms of the Gaussian Fibonacci sequence are , $1, 1 + i, 2 + i, 3 + 2i, 5 + 3i, 8 + 5i, \ldots$. It can be easily verified that the Binet-type formula for GF_n is given by $GF_n = \left(\frac{\alpha^n - \beta^n}{\alpha-\beta}\right) + i\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha-\beta}\right)$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The Gaussian Lucas sequence $\{GL_n\}$ is defined by the recurrence relation $GL_n = GL_{n-1} + GL_{n-2}$, for all $n \geq 2$ with initial conditions $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$. The first few terms of the Gaussian Lucas sequence are $2 - i, 1 + 2i, 3 + i, 4 + 3i, 7 + 4i, 11 + 7i, \ldots$. The Binet-type formula for GL_n can be derived as $GL_n = (\alpha^n + \beta^n) + i(\alpha^{n-1} + \beta^{n-1})$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The Gaussian Pell sequence is defined by the recurrence relation $GP_n = 2GP_{n-1} + GP_{n-2}$, for all $n \geq 2$ with initial conditions $GP_0 = i$ and $GP_1 = 1$. The first few terms of the Gaussian Pell sequence are $1, 2 + i, 5 +$

$2i, 12+5i, \dots$. The Binet-type formula for GP_n is given by $GP_n = \left(\frac{\gamma^n - \delta^n}{\gamma-\delta}\right) + i\left(\frac{\gamma^{n-1} - \delta^{n-1}}{\gamma-\delta}\right)$, where $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$.

The Gaussian Pell-Lucas sequence is defined by the recurrence relation $GQ_n = 2GQ_{n-1} + GQ_{n-2}$, for all $n \geq 2$ with initial conditions $GQ_0 = 2-2i$ and $GQ_1 = 2+2i$. The first few terms of the Gaussian Pell-Lucas sequence are $2-2i, 2+2i, 6+2i, 14+6i, \dots$. The Binet-type formula for GQ_n is given by $GQ_n = (\gamma^n + \delta^n) + i(\gamma^{n-1} + \delta^{n-1})$, where $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$.

II. AREA UNDER THE BINET-CURVE FOR $GF_t^{L(a,b)}$

In the Binet-type formula for $GF_n^{L(a,b)}$, we replace the integer index *n* by some variable $t \in \mathbb{R}$. It can be then observed that the corresponding generalized Right k-Fibonacci number $GF_t^{L(a,b)}$ will be convert into a complex number. (See: [2,5]). Thus, the corresponding Binet-type formula can be considered as $F_t^{L(a,b)} = \frac{h\alpha^t - l\beta^t}{\alpha-\beta}$, where t is any arbitrary real number. As noted earlier, we have $\alpha\beta = -1$, Then $\beta^t = \left(\frac{-1}{\alpha}\right)^t = \frac{(-1)^t}{\alpha^t} = \frac{e^{i\pi t}}{\alpha^t}$. Now $F_t^{L(a,b)} = \frac{h\alpha^t - l\left(\frac{-1}{\alpha}\right)^t}{\alpha-\beta}$. Since $h = \alpha a + (b-ka) + i\{a - \beta(b-ka)\}$ and $l = \beta a + (b-ka) + i\{a - \alpha(b-ka)\}$ are complex numbers, we let $h = r+is$ and $l = u+iv$, where $r = \alpha a + (b-ka), s = a - \beta(b-ka)$, $u = \beta a + (b-ka)$ and $v = a - \alpha(b-ka)$.

$$\therefore GF_t^{L(a,b)} = \frac{(r+is)e^{t\,ln(\alpha)} - (u+iv)e^{i\pi t}e^{-t\,ln(\alpha)}}{\alpha-\beta} = \frac{(r+is)e^{t\,ln(\alpha)} - (u+iv)(\cos(\pi t) + i\sin(\pi t))e^{-t\,ln(\alpha)}}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\left\{re^{t\,ln(\alpha)} + e^{-t\,ln(\alpha)}(v\sin(\pi t) - u\cos(\pi t))\right\}$$

$$+ \frac{i}{\alpha-\beta}\left\{se^{t\,ln(\alpha)} - e^{-t\,ln(\alpha)}(v\cos(\pi t) + u\sin(\pi t))\right\}.$$

This function describes a curve in the complex plane parameterized by real variable *t*. We thus define the "*Binet-curve*" for $-\infty < t < \infty$, in a parametric form as

$$GF_t^{L(a,b)} = (x(t), y(t)); \tag{2}$$

where

$$\left.\begin{aligned} Re\left(GF_t^{L(a,b)}\right) = x(t) = \frac{1}{\alpha-\beta}\left\{re^{t\,ln(\alpha)} + e^{-t\,ln(\alpha)}(v\sin(\pi t) - u\cos(\pi t))\right\} \\ Im\left(GF_t^{L(a,b)}\right) = y(t) = \frac{1}{\alpha-\beta}\left\{se^{t\,ln(\alpha)} - e^{-t\,ln(\alpha)}(v\cos(\pi t) + u\sin(\pi t))\right\} \end{aligned}\right\}. \tag{3}$$

In the following theorem we calculate the area under the Binet-curve for $GF_t^{L(a,b)}$ for any arbitrary real number t within the fixed interval.

Theorem 2.1: Area of the segment under the Binet-curve for $GF_t^{L(a,b)}$ within the interval $[n, n+1]$ is given by

$$A_{n,n+1} = \frac{1}{(\alpha-\beta)^2}\left[\frac{4sv\,ln(\alpha)(-1)^n}{\pi} + 2(rv - su)(-1)^n + \frac{(v^2+u^2)\pi}{ln(\alpha)} \cdot \frac{1}{\alpha^{2n+1}}\right] \tag{4}$$

Proof: We use the Green's theorem to calculate the area value of segments $A_{n,n+1} = \frac{1}{2}\int_n^{n+1}(x\,dy - y\,dx)$. By rewriting $dy = \left(\frac{dy}{dt}\right)dt$ and $dx = \left(\frac{dx}{dt}\right)dt$, the formula becomes

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1}(x\,\dot{y} - y\dot{x})dt. \tag{5}$$

We use components $x(t)$ and $y(t)$ of Binet type formula for real argument and calculate this value. Now,

$$\dot{x} = \frac{dx}{dt} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} rln(\alpha)e^{t\,ln(\alpha)} - ln(\alpha)\,e^{-t\,ln(\alpha)}(v\sin(\pi t) - u\cos(\pi t)) \\ +e^{-t\,ln(\alpha)}(v\pi\cos(\pi t) + u\pi\sin(\pi t)) \end{matrix}\right\} \text{ and}$$

$$\dot{y} = \frac{dy}{dt} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} sln(\alpha)e^{t\,ln(\alpha)} + ln(\alpha)\,e^{-t\,ln(\alpha)}(v\cos(\pi t) + u\sin(\pi t)) \\ -e^{-t\,ln(\alpha)}(-v\pi\sin(\pi t) + u\pi\cos(\pi t)) \end{matrix}\right\}.$$

$$\text{Thus } x\,\dot{y} - y\dot{x} = \frac{1}{(\alpha-\beta)^2}\left\{\begin{matrix} 2\,ln(\alpha)\,\{(rv - su)\cos(\pi t) + (ru + vs)\sin(\pi t)\} \\ +\pi(rv - su)\sin(\pi t) - \pi(ru + vs)\cos(\pi t) \\ +e^{-2t\,ln(\alpha)}(v^2 + u^2)\pi \end{matrix}\right\}.$$

Substituting this value in (5) we get

$$A_{n,n+1} = \frac{1}{2(\alpha-\beta)^2}\int_n^{n+1}\left\{\begin{matrix} 2\,ln(\alpha)\,\{(rv-su)\cos(\pi t)+(ru+vs)\sin(\pi t)\} \\ +\pi(rv-su)\sin(\pi t)-\pi(ru+vs)\cos(\pi t) \\ +e^{-2t\,ln(\alpha)}(v^2+u^2)\pi \end{matrix}\right\}dt$$

$$= \frac{1}{2(\alpha-\beta)^2}\left\{\begin{matrix} 2\,ln(\alpha)\,(rv-su)\left[\frac{\sin(\pi t)}{\pi}\right]_n^{n+1}+2\,ln(\alpha)\,(ru+vs)\left[-\frac{\cos(\pi t)}{\pi}\right]_n^{n+1} \\ +\pi(rv-su)\left[-\frac{\cos(\pi t)}{\pi}\right]_n^{n+1}-\pi(ru+vs)\left[\frac{\sin(\pi t)}{\pi}\right]_n^{n+1} \\ +(v^2+u^2)\pi\left[\frac{e^{-2t\,ln(\alpha)}}{-2\,ln(\alpha)}\right]_n^{n+1} \end{matrix}\right\}$$

Hence, $A_{n,n+1} = \frac{1}{2(\alpha-\beta)^2}\left\{\frac{4\,ln(\alpha)(ru+vs)(-1)^n}{\pi}+2(rv-su)(-1)^n+\frac{(v^2+u^2)\pi}{ln(\alpha)}\left(\frac{1}{\alpha^{2n+1}}\right)\right\}$.

Remark: If we take limit value of area segment at infinity, we have

$A_\infty = \lim_{n\to\infty}\left|A_{n,n+1}\right| = \frac{1}{2(\alpha-\beta)^2}\left(\frac{4\,ln(\alpha)(ru+vs)}{\pi}+2(rv-su)\right)$.

Thus, we observe that the sequence of segment's area has finite limit.

III. CURVATURE OF THE BINET-CURVE FOR $GF_t^{L(a,b)}$

In this section, we find the value of curvature $\kappa(t)$ of the Binet-curve for $GF_t^{L(a,b)}$ at any arbitrary point $\left(x(t), y(t)\right)$.

Theorem 3.1: The curvature of the Binet-curve $GF_t^{L(a,b)}$ at any arbitrary point is

$$\kappa(t) = \frac{\sqrt{k^2+4}\begin{pmatrix} -(ln(\alpha))^2\{v\cos(\pi t)+u\sin(\pi t)\}+(1+2\pi)(ln(\alpha))^2\{(sv+ru)\cos(\pi t)+(su-rv)\sin(\pi t)\} \\ +\pi^2\,ln(\alpha)\{(rv-su)\cos(\pi t)+(ru+sv)\sin(\pi t)\} \\ +(ln(\alpha))^3\{(-2su+rv)\cos(\pi t)+(2sv+ru)\sin(\pi t)\} \\ +\{\pi(ln(\alpha))^2+\pi^3\}e^{-2t\,ln(\alpha)}(v^2+u^2)-ln(\alpha)e^{-2t\,ln(\alpha)}\pi^2(2u\sin(\pi t))(u\cos(\pi t)-v\sin(\pi t)) \end{pmatrix}}{\begin{pmatrix} (r^2+s^2)(ln(\alpha))^2e^{2t\,ln(\alpha)}+((ln(\alpha))^2+\pi)(v^2+u^2)e^{2t\,ln(\alpha)} \\ +2(ln(\alpha))^2\{(sv+ru)\cos(\pi t)+(su-rv)\sin(\pi t)\} \\ +2\pi\,ln(\alpha)\{(rv-su)\cos(\pi t)+(ru+sv)\sin(\pi t)\} \end{pmatrix}^{3/2}}. \quad (6)$$

Proof: As discussed earlier, we consider

$$\vec{r}(t) = GF_n^{L(a,b)} = \left(x(t), y(t)\right) = \left(\frac{1}{\alpha-\beta}\left\{re^{t\,ln(\alpha)}+e^{-t\,ln(\alpha)}(v\sin(\pi t)-u\cos(\pi t))\right\},\right.$$
$$\left.\frac{1}{\alpha-\beta}\left\{se^{t\,ln(\alpha)}-e^{-t\,ln(\alpha)}(v\cos(\pi t)+u\sin(\pi t))\right\}\right)$$

We use the curvature formula $\kappa(t) = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$.

Now it can be shown that

$$\dot{x} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} rln(\alpha)e^{t\,ln(\alpha)}-ln(\alpha)\,e^{-t\,ln(\alpha)}(v\sin(\pi t)-u\cos(\pi t)) \\ +e^{-t\,ln(\alpha)}(v\pi\cos(\pi t)+u\pi\sin(\pi t)) \end{matrix}\right\},$$

$$\ddot{x} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} r(ln\,\alpha)^2e^{t\,ln(\alpha)}+(ln\,\alpha)^2e^{-t\,ln(\alpha)}(v\sin(\pi t)-u\cos(\pi t)) \\ -2\,ln(\alpha)\,e^{-t\,ln(\alpha)}(v\pi\cos(\pi t)+u\pi\sin(\pi t)) \\ +e^{-t\,ln(\alpha)}(-v\pi^2\sin(\pi t)+u\pi^2\cos(\pi t)) \end{matrix}\right\},$$

$$\dot{y} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} sln(\alpha)e^{t\,ln(\alpha)}+ln(\alpha)\,e^{-t\,ln(\alpha)}(v\cos(\pi t)+u\sin(\pi t)) \\ -e^{-t\,ln(\alpha)}(v\pi\sin(\pi t)+u\pi\cos(\pi t)) \end{matrix}\right\} \text{ and}$$

$$\ddot{y} = \frac{1}{\alpha-\beta}\left\{\begin{matrix} s(ln\,\alpha)^2e^{t\,ln(\alpha)}-(ln\,\alpha)^2e^{-t\,ln(\alpha)}(v\cos(\pi t)+u\sin(\pi t)) \\ +2\,ln(\alpha)\,e^{-t\,ln(\alpha)}(-\,v\,\pi\sin(\pi t)+u\pi\cos(\pi t)) \\ -e^{-t\,ln(\alpha)}(-v\pi^2\cos(\pi t)-u\pi^2\sin(\pi t)) \end{matrix}\right\}.$$

Then $|\dot{x}\ddot{y}-\dot{y}\ddot{x}| =$

$$\frac{1}{5}\begin{pmatrix} -(ln(\alpha))^2\{v\cos(\pi t)+u\sin(\pi t)\}+(1+2\pi)(ln(\alpha))^2\{(sv+ru)\cos(\pi t)+(su-rv)\sin(\pi t)\} \\ +\pi^2\,ln(\alpha)\,\{(rv-su)\cos(\pi t)+(ru+sv)\sin(\pi t)\} \\ +(ln(\alpha))^3\{(-2su+rv)\cos(\pi t)+(2sv+ru)\sin(\pi t)\} \\ +\{\pi(ln(\alpha))^2+\pi^3\}e^{-2t\,ln(\alpha)}(v^2+u^2)-ln(\alpha)\,e^{-2t\,ln(\alpha)}\pi^2(2u\sin(\pi t))(u\cos(\pi t)-v\sin(\pi t)) \end{pmatrix}.$$

Also,

$\dot{x}^2 =$

$$\frac{1}{5}\begin{pmatrix} r^2(\ln\alpha)^2e^{2t\ln(\alpha)} + (\ln\alpha)^2e^{-2t\ln(\alpha)}(v\sin(\pi t) - u\cos(\pi t))^2 \\ +e^{-2t\ln(\alpha)}(v\pi\cos(\pi t) + u\pi\sin(\pi t))^2 - 2r(\ln\alpha)^2(v\sin(\pi t) - u\cos(\pi t)) \\ +2r\ln\alpha\,(v\pi\cos(\pi t) + u\pi\sin(\pi t)) - \ln\alpha\, e^{-2t\ln(\alpha)}(v\sin(\pi t) - u\cos(\pi t))(v\pi\cos(\pi t) + u\pi\sin(\pi t)) \end{pmatrix}$$

and

$\dot{y}^2 =$

$$\frac{1}{5}\begin{pmatrix} g^2(\ln\alpha)^2e^{2t\ln(\alpha)} + (\ln\alpha)^2e^{-2t\ln(\alpha)}(v\cos(\pi t) + u\sin(\pi t))^2 \\ +e^{-2t\ln(\alpha)}(-v\pi\sin(\pi t) + u\pi\cos(\pi t))^2 - 2g(\ln\alpha)^2(v\cos(\pi t) + u\sin(\pi t)) \\ -2g\ln\alpha\,(-v\pi\sin(\pi t) + u\pi\cos(\pi t)) - \ln\alpha\, e^{-2t\ln(\alpha)}(v\cos(\pi t) + u\sin(\pi t))(-v\pi\sin(\pi t) + u\pi\cos(\pi t)) \end{pmatrix}$$

Then

$$\dot{x}^2 + \dot{y}^2 = \frac{1}{5}\begin{pmatrix} (r^2+s^2)(\ln(\alpha))^2e^{2t\ln(\alpha)} + ((\ln(\alpha))^2+\pi)(v^2+u^2)e^{2t\ln(\alpha)} \\ +2(\ln(\alpha))^2\{(sv+ru)\cos(\pi t) + (su-rv)\sin(\pi t)\} \\ +2\pi\ln(\alpha)\{(rv-su)\cos(\pi t) + (ru+sv)\sin(\pi t)\} \end{pmatrix}.$$

Using these values, we finally get the value of $\kappa(t)$ as follows:

$$\kappa(t) = \frac{\sqrt{k^2+4}\begin{pmatrix} -(\ln(\alpha))^2\{v\cos(\pi t)+u\sin(\pi t)\}+(1+2\pi)(\ln(\alpha))^2\{(sv+ru)\cos(\pi t)+(su-rv)\sin(\pi t)\} \\ +\pi^2\ln(\alpha)\{(rv-su)\cos(\pi t)+(ru+sv)\sin(\pi t)\} \\ +(\ln(\alpha))^3\{(-2su+rv)\cos(\pi t)+(2sv+ru)\sin(\pi t)\} \\ +\{\pi(\ln(\alpha))^2+\pi^3\}e^{-2t\ln(\alpha)}(v^2+u^2)-\ln(\alpha)e^{-2t\ln(\alpha)}\pi^2(2u\sin(\pi t))(u\cos(\pi t)-v\sin(\pi t)) \end{pmatrix}}{\begin{pmatrix} (r^2+s^2)(\ln(\alpha))^2e^{2t\ln(\alpha)}+((\ln(\alpha))^2+\pi)(v^2+u^2)e^{2t\ln(\alpha)} \\ +2(\ln(\alpha))^2\{(sv+ru)\cos(\pi t)+(su-rv)\sin(\pi t)\} \\ +2\pi\ln(\alpha)\{(rv-su)\cos(\pi t)+(ru+sv)\sin(\pi t)\} \end{pmatrix}^{3/2}}.$$

Note: By taking limit as $t \to \pm\infty$, we finally get $\lim_{t\to\infty}\kappa(t) = 0$ and $\lim_{t\to-\infty}\kappa(t) = 0$. Thus, the curvature is zero at any point of the curve $GF_t^{L(a,b)}$, when t is very large. This means the curve behaves like a straight line at distant points of $GF_t^{L(a,b)}$.

In the following sections, we choose different values of k, a, b and completely analyze and trace the Binet-type curve for these sequences.

IV. BINET-CURVE FOR GAUSSIAN FIBONACCI SEQUENCE AND ITS SUBSEQUENCE

In this section, we consider $k = 1$ and $a = 0, b = 1$ in $GF_n^{L(a,b)}$. Then we get the sequence $\left\{GF_n^{L(0,1)}\right\} = \{GF_n\}$ of *Gaussian Fibonacci numbers* from (1). Considering $k = 1, a = 0, b = 1$ in (2) and (3), we get "*Gaussian Fibonacci - Binet curve*" for $-\infty < t < \infty$, in a parametric form as $GF_t = (x(t), y(t))$. Then

$$Re(GF_t) = x(t) = \frac{1}{\alpha-\beta}\left\{e^{t\ln(\alpha)} - e^{-t\ln(\alpha)}(\cos(\pi t) + \alpha\sin(\pi t))\right\} \text{ and}$$

$$Im(GF_t) = y(t) = \frac{1}{\alpha-\beta}\left\{\alpha^{-1}e^{t\ln(\alpha)} + e^{-t\ln(\alpha)}\big(\alpha\cos(\pi t) - \sin(\pi t)\big)\right\}.$$

Now, we analyze the parametric curve $(x(t),\ y(t))$ of Gaussian Fibonacci numbers in connection with its symmetry, origin, intersection of points, and asymptotes etc., and we conclude the following:

(i) The curve is not symmetric about x, y axes, since x, y are not even functions of t.

(ii) The curve passes through the origin, since there exists $t = 0$ at which $x = 0$ and $y = 0$.

(iii) The curve intersects x-axis at the point $(1, 0)$ and intersects y-axis at the point $(0,1)$.

(iv) When we consider

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{5}}\{\alpha^{-1}\ln(\alpha)e^{t\ln(\alpha)} - \ln(\alpha)e^{-t\ln(\alpha)}(\alpha\cos(\pi t) - \sin(\pi t)) + e^{-t\ln(\alpha)}(-\alpha\pi\sin(\pi t) - \pi\cos(\pi t))\}}{\frac{1}{\sqrt{5}}\{\ln(\alpha)e^{t\ln(\alpha)} + \ln(\alpha)e^{-t\ln(\alpha)}(\cos(\pi t) + \alpha\sin(\pi t)) - e^{-t\ln(\alpha)}(-\pi\sin(\pi t) + \alpha\pi\cos(\pi t))\}} = 0,$$

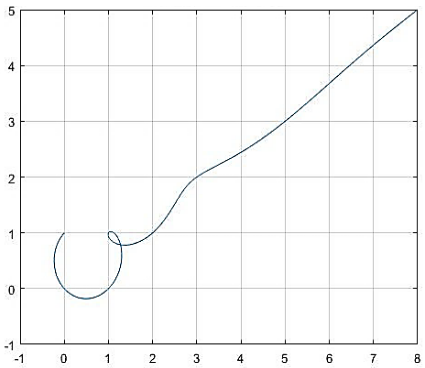
we have $\alpha^{-1}\ln(\alpha)\,e^{t\ln(\alpha)} - \ln(\alpha)\,e^{-t\ln(\alpha)}(\alpha\cos(\pi t) - \sin(\pi t)) + e^{-t\ln(\alpha)}(-\alpha\pi\sin(\pi t) - \pi\cos(\pi t)) = 0$. This gives $\alpha^{-1}\ln(\alpha)\,e^{t\ln(\alpha)} = \{\cos(\pi t)\,(\alpha\ln(\alpha) + \pi) + \sin(\pi t)\,(\ln(\alpha) - \alpha\pi)\}e^{-t\ln(\alpha)}$. Then $e^{2t\ln(\alpha)} = \cos(\pi t)\left(\frac{\alpha\ln(\alpha)+\pi}{\alpha^{-1}\ln(\alpha)}\right) + \sin(\pi t)\left(\frac{\ln(\alpha)-\alpha\pi}{\alpha^{-1}\ln(\alpha)}\right)$. Thus, using the software MATLAB, we have $t = 0.22266$, where $\alpha = 1.618$. Hence, we conclude that x-axis is the tangent to the curve at $t = 0.22266$.

(v) Using Theorem 3.1, we observe that the curvature for the Gaussian Fibonacci-Binet curve will be

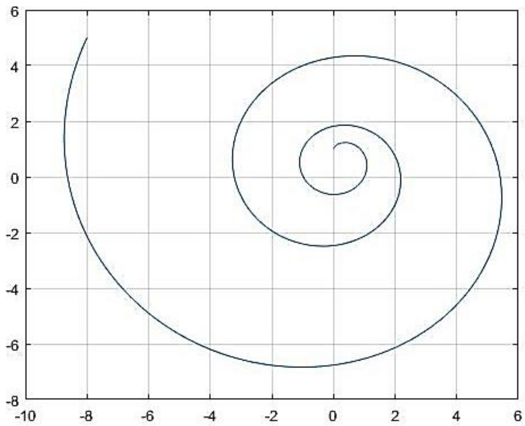
$$\kappa(t)=\frac{\begin{pmatrix}\{2(ln(\alpha))^3-3\alpha^{-1}\pi(ln(\alpha))^2-\pi^2\,ln(\alpha)\}(\alpha\cos(\pi t)-\sin(\pi t))\\+\{3\pi(ln(\alpha))^2-\alpha^{-1}\pi^2\,ln(\alpha)+2\alpha^{-1}(ln(\alpha))^3\}(\cos(\pi t)+\alpha\sin(\pi t))\\+\{(ln(\alpha))^2+\pi^2\}(\alpha^2+1)\pi e^{-2t\,ln(\alpha)}\end{pmatrix}}{\begin{pmatrix}\{(ln(\alpha))^2+\pi^2\}(\alpha^2+1)e^{-2t\,ln(\alpha)}+(1+\alpha^{-2})(ln(\alpha))^2e^{2t\,ln(\alpha)}\\+2\,ln(\alpha)\{ln(\alpha)-\alpha^{-1}\}(\cos(\pi t)+\alpha\sin(\pi t))-2\,ln(\alpha)(\pi+\alpha^{-1}\,ln(\alpha))(\alpha\cos(\pi t)-\sin(\pi t))\end{pmatrix}^{3/2}}.$$

Note: By taking limit as $t\to\pm\infty$, we get $\lim_{t\to\pm\infty}\kappa(t)=0$. Thus, at Gaussian Fibonacci number points the curvature of this curve is 0, when t is close to $\pm\infty$. This means the curve behaves like a straight line at distant Gaussian Fibonacci number points.

On the basis of these data, the Gaussian Fibonacci curve can be traced using the software MATLAB as shown in Figure 1, when $t>0$; as well as in Figure 2, when $t<0$.



(Figure 1 Gaussian Fibonacci-Binet curve for $t>0$)



(Figure 2 Gaussian Fibonacci-Binet spiral for $t<0$)

In contrast to positive t, for negative t we get a spiral crossing real line from both sides. We call such curve as "*Gaussian Fibonacci-Binet Spiral curve*" since it intersects real axis at distant Gaussian Fibonacci numbers.

Next, we mention the area under the Gaussian Fibonacci-Binet curve.

Theorem 4. 1: Area of the segment under the Gaussian Fibonacci-Binet curve within the interval $[n,n+1]$ is $A_{n,n+1}=\frac{1}{10}\left[-2(\alpha+\alpha^{-1})(-1)^n+\frac{(\alpha^2+1)\pi}{2\,ln(\alpha)}\cdot\frac{1}{\alpha^{2n+1}}\right]$.

Remark: If we take limit value of area segment at infinity, we have $A_\infty=\lim_{n\to\infty}\left|A_{n,n+1}\right|=\frac{1}{5}(\alpha+\alpha^{-1})$.

V. BINET-CURVE FOR GAUSSIAN LUCAS SEQUENCE AND ITS SUBSEQUENCE

In this section, we consider $k=1$ and $a=2,b=1$ in $GF_n^{L(a,b)}$. Then we get the sequence $\left\{GF_n^{L(2,1)}\right\}=\{GL_n\}$ of Gaussian Lucas sequence from the equation (1). Considering $k=1,a=2,b=1$ in (2) and (3), we get "*Gaussian Lucas-Binet curve*" for $-\infty<t<\infty$, in a parametric form as $GL_t=(x(t),y(t))$; then

$$Re(GL_t)=x(t)=e^{t\,ln(\alpha)}+e^{-t\,ln(\alpha)}(\cos(\pi t)+\alpha\sin(\pi t)),$$
$$Im(GL_t)=y(t)=\alpha^{-1}e^{t\,ln(\alpha)}+e^{-t\,ln(\alpha)}(\sin(\pi t)-\alpha\cos(\pi t)).$$

Now, we analyze the parametric curve $(x(t),y(t))$ of Gaussian Lucas numbers.

(i) The curve is not symmetric about x,y axes, since x,y are not even functions of t.

(ii) The curve does not pass through the origin, since there exists any real value of t at which $x=0$ and $y=0$.

(iii) When we consider $\frac{dy}{dx}=\frac{\alpha^{-1}\,ln(\alpha)e^{t\,ln(\alpha)}-ln(\alpha)e^{-t\,ln(\alpha)}(\sin(\pi t)-\alpha\cos(\pi t))+e^{-t\,ln(\alpha)}(\pi\cos(\pi t)+\alpha\pi\sin(\pi t))}{ln(\alpha)e^{t\,ln(\alpha)}-ln(\alpha)e^{-t\,ln(\alpha)}(\cos(\pi t)+\alpha\sin(\pi t))+e^{-t\,ln(\alpha)}(-\pi\sin(\pi t)+\alpha\pi\cos(\pi t))}=0$, we have $\alpha^{-1}\,ln(\alpha)\,e^{t\,ln(\alpha)}-ln(\alpha)\,e^{-t\,ln(\alpha)}(\sin(\pi t)-\alpha\cos(\pi t))+e^{-t\,ln(\alpha)}(\pi\cos(\pi t)+\alpha\pi\sin(\pi t))=0$. This gives $\alpha^{-1}\,ln(\alpha)\,e^{t\,ln(\alpha)}=\{\sin(\pi t)\,(ln(\alpha)-\alpha\pi)-\cos(\pi t)\,(\alpha\,ln(\alpha)+\pi)\}e^{-t\,ln(\alpha)}$. Then $e^{2t\,ln(\alpha)}=$

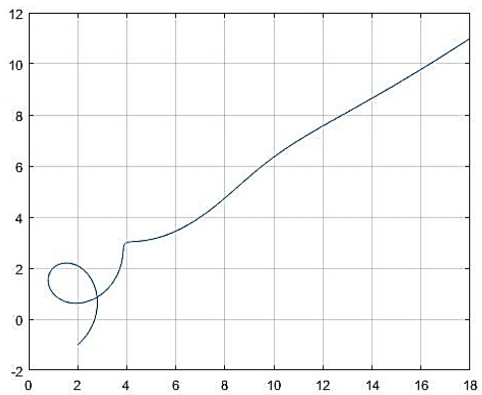
$\sin(\pi t)\left(\frac{ln(\alpha)-\alpha\pi}{\alpha^{-1}\, ln(\alpha)}\right)-\cos(\pi t)\left(\frac{\alpha\, ln(\alpha)+\pi}{\alpha^{-1}\, ln(\alpha)}\right)$. Thus, using software MATLAB, we have $t=-0.225854$, where $\alpha=1.618$. Hence, we conclude that x-axis is the tangent to the curve at the value $t=-0.225854$.

(iv) Using Theorem (3.1), we observe that the curvature for the Gaussian Lucas-Binet curve will be

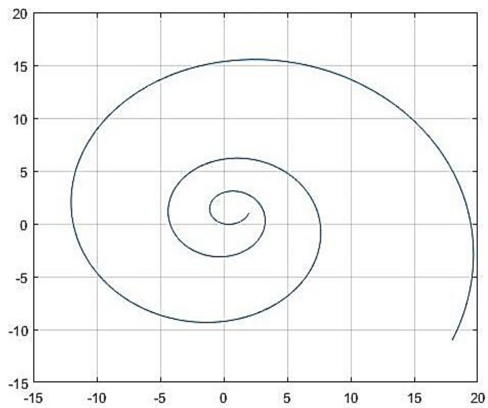
$$\kappa(t)=\frac{\begin{pmatrix}\{2(ln(\alpha))^3-3\alpha^{-1}\pi(ln(\alpha))^2-\pi^2\, ln(\alpha)\}(\sin(\pi t)-\alpha\cos(\pi t))\\ +\{3\pi(ln(\alpha))^2+\alpha^{-1}\pi^2\, ln(\alpha)-2\alpha^{-1}(ln(\alpha))^3\}(\cos(\pi t)+\alpha\sin(\pi t))\\ +\{(ln(\alpha))^2+\pi^2\}(\alpha^2+1)\pi e^{-2t\, ln(\alpha)}\end{pmatrix}}{\begin{pmatrix}\{(ln(\alpha))^2+\pi^2\}(\alpha^2+1)e^{-2t\, ln(\alpha)}+(1+\alpha^{-2})(ln(\alpha))^2e^{2t\, ln(\alpha)}\\ +2\, ln(\alpha)\{\alpha^{-1}-ln(\alpha)\}(\cos(\pi t)+\alpha\sin(\pi t))-2\, ln(\alpha)(\pi+\alpha^{-1}\, ln(\alpha))(\sin(\pi t)-\alpha\cos(\pi t))\end{pmatrix}^{3/2}}.$$

Note: By taking limit as $t\to\pm\infty$, we get $\lim_{t\to\pm\infty}\kappa(t)=0$. Thus, at Gaussian Lucas number points the curvature of this curve is 0, when t is close to $\pm\infty$. This means the curve behaves like a straight line at distant Gaussian Lucas number points.

On the basis of these data, the Gaussian Lucas curve can be traced using the software MATLAB as shown in Figure 3, when $t>0$; as well as in Figure 4, when $t<0$.



(Figure 3 Gaussian Lucas-Binet curve for $t>0$)



(Figure 4 Gaussian Lucas-Binet spiral for $t<0$)

In contrast to positive t, for negative t we get a spiral crossing real line from both sides. We call such curve as "*Gaussian Lucas-Binet Spiral curve*" since it intersects real axis at Gaussian Lucas numbers.

Next, we mention the area under the Gaussian Lucas-Binet curve.

Theorem 5.1.: Area of the segment under the Gaussian Lucas-Binet curve within the interval $[n,n+1]$ is $A_{n,n+1}=\frac{1}{2}\left[2\alpha(-1)^n+\frac{(\alpha^2+1)\pi}{2\, ln(\alpha)}\cdot\frac{1}{\alpha^{2n+1}}\right]$.

Remark: If we take limit value of area segment at infinity, we have $A_\infty=\lim_{n\to\infty}|A_{n,n+1}|=\alpha$.

VI. BINET-CURVE FOR GAUSSIAN PELL SEQUENCE AND ITS SUBSEQUENCE

If we consider $k=2$ and $a=0, b=1$ in $GF_n^{L(a,b)}$, then we get the sequence $\left\{GF_n^{L(0,1)}\right\}=\{GP_n\}$ of Gaussian Pell sequence from the equation (1). Considering $k=2, a=0, b=1$ in (2) and (3), we get "*Gaussian Pell - Binet curve*" for $-\infty<t<\infty$, in a parametric form as $GP_t=(x(t),y(t))$. Then

$$Re(GP_t)=x(t)=\frac{1}{\gamma-\delta}\{e^{t\, ln(\gamma)}-e^{-t\, ln(\gamma)}(\cos(\pi t)+\gamma\sin(\pi t))\}$$

$$Im(GP_t)=y(t)=\frac{1}{\gamma-\delta}\{\gamma^{-1}e^{t\, ln(\gamma)}+e^{-t\, ln(\gamma)}(\gamma\cos(\pi t)-\sin(\pi t))\}.$$

When we analyze the parametric curve $(x(t),\, y(t))$ of Gaussian Pell numbers, we conclude the following:

(i) The curve is not symmetric about x, y axes, since x, y are not even functions of t.

(ii) The curve passes through the origin, since there exists $t=0$ at which $x=0$ and $y=0$.

(iii) The curve intersects x-axis at the points $(0,1)$ and intersects y-axis at the points $(1,0)$.

(iv) When we consider

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{2}}\{\gamma^{-1}\, ln(\gamma) e^{t\, ln(\gamma)} - ln(\gamma) e^{-t\, ln(\gamma)}(\gamma\cos(\pi t) - \sin(\pi t)) + e^{-t\, ln(\gamma)}(-\pi\cos(\pi t) - \gamma\pi\sin(\pi t))\}}{\frac{1}{2\sqrt{2}}\{ln(\gamma) e^{t\, ln(\gamma)} + ln(\gamma) e^{-t\, ln(\gamma)}(\cos(\pi t) + \gamma\sin(\pi t)) - e^{-t\, ln(\gamma)}(-\pi\sin(\pi t) + \gamma\pi\cos(\pi t))\}} = 0,$$

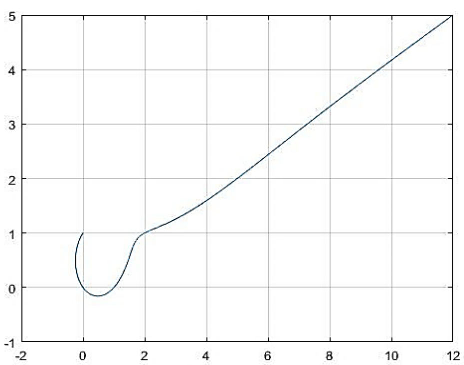
we have $\gamma^{-1}\, ln(\gamma)\, e^{t\, ln(\gamma)} - ln(\gamma)\, e^{-t\, ln(\gamma)}(\gamma\cos(\pi t) - \sin(\pi t)) + e^{-t\, ln(\gamma)}(-\pi\cos(\pi t) - \gamma\pi\sin(\pi t)) = 0$. This gives $\gamma^{-1}\, ln(\gamma)\, e^{t\, ln(\gamma)} = \{\cos(\pi t)\,(\gamma\, ln(\gamma) + \pi) + \sin(\pi t)\,(\gamma\pi - ln(\gamma))\} e^{-t\, ln(\gamma)}$. Then $e^{2t\, ln(\gamma)} = \cos(\pi t)\left(\frac{\gamma\, ln(\gamma) + \pi}{\gamma^{-1}\, ln(\gamma)}\right) + \sin(\pi t)\left(\frac{\gamma\pi - ln(\gamma)}{\gamma^{-1}\, ln(\gamma)}\right)$. Thus, using the software MATLAB, we have $t = -0.202524$, where $\gamma = 2.4142$. Hence, we conclude that x-axis is the tangent to the curve at the value $t = -0.202524$.

(v) Using Theorem (3.1), we observe that the curvature for the Gaussian Pell-Binet curve will be

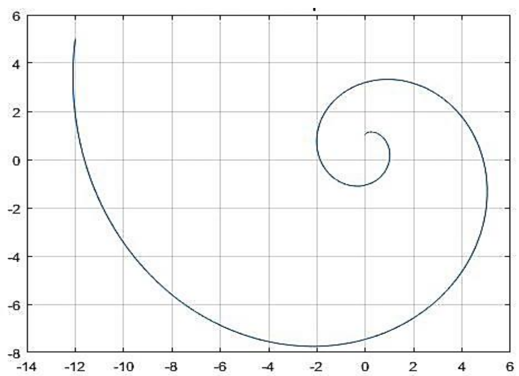
$$\kappa(t) = \frac{2\sqrt{2}\begin{pmatrix} \{2(ln(\gamma))^3 - 3\gamma^{-1}\pi(ln(\gamma))^2 - \pi^2\, ln(\gamma)\}(\gamma\cos(\pi t) - \sin(\pi t)) \\ + \{3\pi(ln(\gamma))^2 - \gamma^{-1}\pi^2\, ln(\gamma) + 2\gamma^{-1}(ln(\gamma))^3\}(\cos(\pi t) + \gamma\sin(\pi t)) \\ + \{(ln(\gamma))^2 + \pi^2\}(\gamma^2 + 1)\pi e^{-2t\, ln(\gamma)} \end{pmatrix}}{\begin{pmatrix} \{(ln(\gamma))^2 + \pi^2\}(\gamma^2 + 1) e^{-2t\, ln(\gamma)} + (1 + \gamma^{-2})(ln(\gamma))^2 e^{2t\, ln(\gamma)} \\ + 2\, ln(\gamma)\{ln(\gamma) - \gamma^{-1}\}(\cos(\pi t) + \gamma\sin(\pi t)) - 2\, ln(\gamma)(\pi + \gamma^{-1}\, ln(\gamma))(\gamma\cos(\pi t) - \sin(\pi t)) \end{pmatrix}^{3/2}}.$$

Note: By taking limit as $t \to \pm\infty$, we get $\lim_{t\to\pm\infty} \kappa(t) = 0$. Thus, at Gaussian Pell number points the curvature of this curve is 0, when t is close to $\pm\infty$. This means the curve behaves like a straight line at distant Gaussian Pell number points.

On the basis of these data, the Gaussian Pell curve can be traced using the software MATLAB as shown in Figure 5, when $t > 0$; as well as in Figure 6, when $t < 0$.



(Figure 5 Gaussian Pell-Binet curve for $t > 0$)



(Figure 6 Gaussian Pell-Binet spiral for $t < 0$)

In contrast to positive t, for negative t we get a spiral crossing real line from both sides. We call such curve as "*Gaussian Pell-Binet Spiral curve*" since it intersects real axis at Gaussian Pell numbers.

We mention the area under the Gaussian Pell-Binet curve.

Theorem 6.1: Area of the segment under the Gaussian Pell-Binet curve within the interval $[n, n+1]$ is $A_{n,n+1} = \frac{1}{10}\left[-2\gamma(-1)^n + \frac{(\gamma^2+1)\pi}{ln(\gamma)} \cdot \frac{1}{\gamma^{2n+1}}\right]$.

Remark: If we take limit value of area segment at infinity, we have $A_\infty = \lim_{n\to\infty}|A_{n,n+1}| = \frac{1}{5}\gamma$

VII. BINET-CURVE FOR GAUSSIAN PELL-LUCAS SEQUENCE AND ITS SUBSEQUENCE

If we consider $k = 2$ and $a = 2, b = 2$ in $GF_n^{L(a,b)}$, then we get the sequence $\left\{GF_n^{L(2,2)}\right\} = \{GQ_n\}$ of Gaussian Pell-Lucas sequence from (1). Considering $k = 2, a = 2, b = 2$ in (2) and (3), we get "*Gaussian Pell-Lucas - Binet curve*" for $-\infty < t < \infty$, in a parametric form as $GQ_t = (x(t), y(t))$. Then

$$Re(GQ_t) = x(t) = e^{t\, ln(\gamma)} + e^{-t\, ln(\gamma)}(\cos(\pi t) + \gamma\sin(\pi t))$$

$$Im(GQ_t) = y(t) = \gamma^{-1} e^{t\, ln(\gamma)} + e^{-t\, ln(\gamma)}(\sin(\pi t) - \gamma\cos(\pi t)).$$

Now, we analyze the parametric curve $(x(t),\ y(t))$ of Gaussian Pell-Lucas numbers.

(i) The curve is not symmetric about x, y axes, since x, y are not even functions of t.

(ii) The curve does not pass through the origin, since there exists any real value of t at which $x = 0$ and $y = 0$.

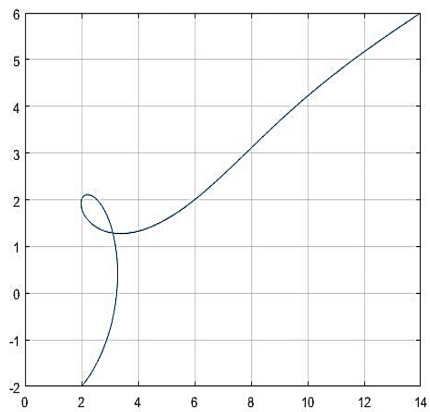
(iii) When we consider $\frac{dy}{dx} = \frac{\gamma^{-1} ln(\gamma)e^{t\, ln(\gamma)} + ln(\gamma)e^{-t\, ln(\gamma)}(\gamma\cos(\pi t) - \sin(\pi t)) + e^{-t\, ln(\gamma)}(\pi\cos(\pi t) + \gamma\pi\sin(\pi t))}{ln(\gamma)e^{t\, ln(\gamma)} - ln(\gamma)e^{-t\, ln(\gamma)}(\cos(\pi t) + \gamma\sin(\pi t)) - e^{-t\, ln(\gamma)}(-\pi\sin(\pi t) + \gamma\pi\cos(\pi t))} = 0$, then we have $\gamma^{-1}\, ln(\gamma)\, e^{t\, ln(\gamma)} + ln(\gamma)\, e^{-t\, ln(\gamma)}(\gamma\cos(\pi t) - \sin(\pi t)) + e^{-t\, ln(\gamma)}(\pi\cos(\pi t) + \gamma\pi\sin(\pi t)) = 0$. This gives $\gamma^{-1}\, ln(\gamma)\, e^{t\, ln(\gamma)} = \{\cos(\pi t)\,(-\gamma\, ln(\gamma) - \pi) + \sin(\pi t)\,(ln(\gamma) - \gamma\pi)\}e^{-t\, ln(\gamma)}$. Then $e^{2t\, ln(\gamma)} = \sin(\pi t)\left(\frac{\gamma\pi - ln(\gamma)}{\gamma^{-1}\, ln(\gamma)}\right) - \cos(\pi t)\left(\frac{\gamma\, ln(\gamma) + \pi}{\gamma^{-1}\, ln(\gamma)}\right)$. Thus, using the software MATLAB, we have $t = -0.371997$, where $\gamma = 2.4142$. Hence, we conclude that x-axis is the tangent to the curve at the value $t = -0.371997$.

(iv) Using (4.3.3), we observe that the curvature for the Gaussian Pell-Lucas-Binet curve will be

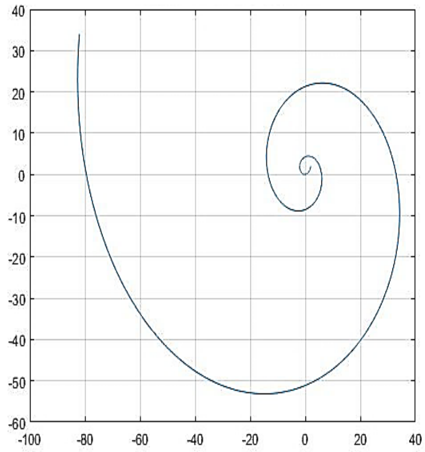
$$\kappa(t) = \frac{\begin{pmatrix} \{2(ln(\gamma))^3 - 3\gamma^{-1}\pi(ln(\gamma))^2 - \pi^2\, ln(\gamma)\}(\sin(\pi t) - \gamma\cos(\pi t)) \\ +\{-3\pi(ln(\gamma))^2 + \gamma^{-1}\pi^2\, ln(\gamma) - 2\gamma^{-1}(ln(\gamma))^3\}(\cos(\pi t) + \gamma\sin(\pi t)) \\ +\{(ln(\gamma))^2 + \pi^2\}(\gamma^2 + 1)\pi e^{-2t\, ln(\gamma)} \end{pmatrix}}{\begin{pmatrix} \{(ln(\gamma))^2 + \pi^2\}(\gamma^2 + 1)e^{-2t\, ln(\gamma)} + (1 + \gamma^{-2})(ln(\gamma))^2 e^{2t\, ln(\gamma)} \\ +2\, ln(\gamma)\{\gamma^{-1} - ln(\gamma)\}(\cos(\pi t) + \gamma\sin(\pi t)) + 2\, ln(\gamma)(\pi - \gamma^{-1}\, ln(\gamma))(\gamma\cos(\pi t) - \sin(\pi t)) \end{pmatrix}^{3/2}}.$$

Note: By taking limit as $t \to \pm\infty$, we get $\lim_{t\to\pm\infty} \kappa(t) = 0$. Thus, at Gaussian Pell-Lucas number points the curvature of this curve is 0, when t is close to $\pm\infty$. This means the curve behaves like a straight line at distant Gaussian Pell-Lucas number points.

On the basis of these data, the Gaussian Pell-Lucas curve can be traced using the software MATLAB as shown in Figure 7, when $t > 0$; as well as in Figure 8, when $t < 0$.



(Figure 7 Gaussian Pell-Lucas-Binet curve for $t > 0$)



(Figure 8 Gaussian Pell-Binet spiral for $t < 0$)

In contrast to positive t, for negative t we get a spiral crossing real line from both sides. We call such curve as "*Gaussian Pell-Lucas-Binet Spiral curve*" since it intersects real axis at Gaussian Pell-Lucas numbers.

We mention the area under the Gaussian Pell-Lucas-Binet curve.

Theorem 7.1: Area of the segment under the Gaussian Pell-Binet curve within the interval $[n, n+1]$ is $A_{n,n+1} = \frac{1}{2}\left[2\gamma(-1)^n + \frac{(\gamma^2+1)\pi}{2\, ln(\gamma)} \cdot \frac{1}{\gamma^{2n+1}}\right]$.

Remark: If we take limit value of area segment at infinity, we have $A_\infty = \lim_{n\to\infty}|A_{n,n+1}| = \gamma$.

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