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A study of Non homogeneous linear differential equation

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Abstract; The differential equation of the form $a_1(x)$ y" + $a_2(x)$ y' + $a_3(x)$ y = f(x) is known as a second-order linear differential equation with variable coefficients. The variable coefficients are $a_1(x)$, $a_2(x)$ and $a_3(x)$
If f(x) = 0 then the above equation is called a homogeneous second-order differential equation.
If f(x) ≠ 0 then the above equation is called a nonhomogeneous second-order differential equation.
In this section give an in depth discussion on the process used to solve homogeneous and non-homogeneous linear, second order differential equations, ay′′+by′+cy=f(x).

Keywords: Second order LDEs with constant coefficient and variable coefficient.

INTRODUCTION

Definition:

The differential equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called linear differential equation where P(x) and Q(x) are function of variable x or in differential equation the dependent variable and its derivatives are not multiplied together.

The differential equation of the form $a_1(x)$ y" + $a_2(x)$ y' + $a_3(x)$ y = f(x) is known as a second-order linear differential equation with variable coefficients. The variable coefficients are $a_1(x)$, $a_2(x)$ and $a_3(x)$

If f(x) = 0 then the above equation is called a homogeneous second-order differential equation.

If f(x) ≠ 0 then the above equation is called a nonhomogeneous second-order differential equation.

where,

$$y'' = \frac{d^2y}{dx^2} and y' = \frac{dy}{dx}$$

For example, y" + y' + 6 = 0 is a second-order linear differential equation with constant coefficient. x^2 y" + 2x y' +y = $4e^x$ is a second-order linear differential equation with variable coefficients.

Nonhomogeneous Differential Equations – In this section we will discuss the basics of solving nonhomogeneous differential equations. We define the complimentary and particular solution and give the form of the general solution to a nonhomogeneous differential equation.

Real Roots – In this section we discuss the solution to second order linear differential equations, ay′′+by′+cy=0, in which the roots of the equation are real distinct roots and real same roots.

Complex Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay′′+by′+cy=0′′+′+=0, in which the roots of equation are complex roots.

Second order differential equation is a specific type of differential equation that consists of a derivative of a function of order 2 and no other higher-order derivative of the function appears in the equation. It includes terms like y'', d^2y/dx^2, y''(x), etc. which indicates the second order derivative of the function. Generally, we write a second order differential equation as y'' + p(x)y' + q(x)y = f(x), where p(x), q(x), and f(x) are functions of x. We can solve this differential equation using the auxiliary equation and different methods such as the method of undetermined coefficients, variation of parameters, by changing dependent and independent variable and exact methods.

The differential equation y'' + p(x)y' + q(x)y = 0 is called a second order differential equation with constant coefficients if the functions p(x) and q(x) are constants and it is called a second-order differential equation with variable coefficients if p(x) and q(x) are not constants. In this article, we will understand such differential equations in detail and their different types. We will also learn different methods to solve second order differential equations with constant coefficients using the various methods with the help of solved examples and finding the auxiliary equation.

What is a Second Order Differential Equation?

A differential equation is an equation that consists of a function and its derivative. A differential equation that consists of a function and its second-order derivative is called a second order differential equation. Mathematically, it is written as y'' + p(x)y' + q(x)y = f(x), which is a non-homogeneous second order differential equation if f(x) is not equal to the zero function and p(x), q(x) are functions of x. It can also be written as F(x, y, y', y'') = 0. Further, let us explore the definitions of the different types of the second order differential equation.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

where P(x),Q(x) and f(x) are functions of x

Second Order Differential Equation Definition

A second order differential equation is defined as a differential equation that includes a function and its second-order derivative and no other higher-order derivative of the function can appear in the equation. It can be of different types depending upon the power of the derivative and the functions involved. These differential equations can be solved using the auxiliary equation. Let us go through some special types of second order differential equations given below:

Linear Second Order Differential Equation

A linear second order differential equation is written as y'' + p(x)y' + q(x)y = f(x), where the power of the second derivative y'' is equal to one which makes the equation linear. Some of its examples are y'' + 4xy =8, x^2 y'' + xy' + y = 0, etc.

Homogeneous Second Order Differential Equation

A second order differential equation y'' + p(x)y' + q(x)y = f(x) is said to be a second order homogeneous differential equation if f(x) is a zero function and hence mathematically it of the form, y'' + p(x)y' + q(x)y = 0. Some of its examples are y'' + y' - 6y = 0, y'' - 9y' + 20y = 0, etc.

Non-homogeneous Second Order Differential Equation

A differential equation of the form y'' + p(x)y' + q(x)y = f(x) is said to be a nonhomogeneous second order differential equation if f(x) is not a zero function. Some of its examples are y'' + y' - 6y = x, y'' - 9y' + 20y = sin x, etc.

Second Order Differential Equation with Constant Coefficients

The differential equation y'' + p(x)y' + q(x)y = f(x) is called a second order differential equation with constant coefficients if the functions p(x) and q(x) are constants. Some of its examples are y'' + y' - 6y = x, y'' - 9y' + 20y = sin x, etc.

Second Order Differential Equation with Variable Coefficients

The differential equation y" + p(x)y' + q(x)y = f(x) is called a second order differential equation with variable coefficients if the functions p(x) and q(x) are not constant functions and are functions of x. Some of its examples are y" + xy' - y sinx = x, $y'' - 9x^2y' + 2e^xy = 0$, etc.

Solving Second Order Differential Equation

We have understood the meaning of second order differential equation and their different forms, we shall proceed towards learning how to solve them. Here, we will focus on learning to solve 2nd differential equations with constant coefficients using the method of undetermined coefficients. First, let us understand how to solve the second order homogeneous differential equations.

Solving Homogeneous Second Order Differential Equation

A homogeneous second order differential equation with constant coefficients is of the form y" + py' + qy = 0, where p, q are constants.

Let us consider a few examples to understand how to determine the solution of the homogeneous second order differential equation with constant coefficient.

Example 1: Solve y'' - 6y' + 5y = 0

Solution: Assume the Characteristic equation

$\Rightarrow r^2 - 6r + 5 = 0 \rightarrow$ Characteristic Equation

$\Rightarrow (r - 5)(r - 1) = 0 \Rightarrow r$

$= 1, 5$

Since the roots of the characteristic equation are distinct and real, therefore the general solution of the given differential equation is $y = c_1 e^x + c_2 e^{5x}$

Example 2: Solve differential equation y" - 8y' + 16y = 0

Solution: Assume Auxiliary Equation

$\Rightarrow m^2 - mr + 16 = 0$

$\Rightarrow (m - 4)(m - 4) = 0 \Rightarrow$

$m = 4, 4$

Since the roots of the characteristic equation are identical and real, therefore the general solution of the given differential equation is

$y = c_1 e^{4x} + c_2 xe^{4x}$

$\Rightarrow y = (c_1 + c_2 x)e^{4x}$

Example 3: Solve differential equation y" +2y' + 2y = 0

Solution: Assume Auxiliary Equation

$\Rightarrow m^2 + 2m + 2 = 0$

$\Rightarrow m = -1 \pm i$

Since the roots of the characteristic equation are complex conjugates, therefore the general solution of the given second order differential equation is $y = e^{-x}[c_1 \sin x + c_2 \cos x]$.

Solving Non-Homogeneous Second Order Differential Equation

To find the solution of Non-Homogeneous Second Order Differential Equation y" + py' + qy = f(x), the general solution is of the form $y = y_c + y_p$, where y_c is the complementary solution of the homogeneous second order differential equation y" + py' + qy = 0 and y_p is the particular solution of the non-homogeneous differential equation y" + py' + qy = f(x). Since y_c is the solution of the homogeneous differential equation, we can determine its value using the methods that we discussed in the previous section. Now, to find the particular solution y_p, we can guess the solution depending upon the value of f(x). The table given below shows the possible particular solution y_p corresponding to each f(x).

In case, f(x) is of a form other than the ones given in the table above, then we can use the method of variation of parameters to solve the non-homogeneous second order differential equation. Also, if f(x) is a sum combination of the functions given in the table, then we can determine the particular solution for each function separately and then take their sum to find the final particular solution of the given equation. Let us now consider a few examples of second order differential equations and solve them using the method of undetermined coefficients:

Example 1: Find the complete solution of the second order differential equation $y'' - 6y' + 5y = e^{-3x}$.
Solution: To find the complete solution, first we will find the general solution of the homogeneous differential equation $y'' - 6y' + 5y = 0$.
We have solved this equation in the previous section in the solved examples (Example 1) and hence the complementary solution is $y_c = c_1 e^x + c_2 e^{5x}$
Next, we will find the particular solution y_p. Since $f(x) = e^{-3x}$
we have
e^{-3x}
$\Rightarrow 9Ae^{-3x} - 6(-3Ae^{-3x}) + 5Ae^{-3x} = e^{-3x}$
$\Rightarrow Ae^{-3x} (9 + 18 + 5) = e^{-3x}$
$\Rightarrow 32 A e^{-3x} = e^{-3x} \Rightarrow A$
$= 1/32$
Hence, the particular solution $y_p = \frac{e^{-3x}}{D^2-6D+5} = \frac{e^{-3x}}{(-3)^2-6(-3)+5} = \frac{e^{-3x}}{32}$
Therefore, the complete solution of the given non-homogeneous differential equation
$y'' - 6y' + 5y = e^{-3x}$ is $y = c_1 e^x + c_2 e^{5x} + (1/32) e^{-3x}$

Example 2: Solve differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$
Solution: As we have solved the homogeneous differential equation $y'' - 6y' + 5y = 0$ in the previous section (Example 1), we have the complementary solution $y_c = c_1 e^x + c_2 e^{5x}$
Next, we will find the particular solution of the given differential equation individually for cos 2x and e^{-3x}, that is, determine the particular solution of $y'' - 6y' + 5y = \cos 2x$ and $y'' - 6y' + 5y = e^{-3x}$ separately. From example 1 above, we have the particular solution of the differential equation $y'' - 6y' + 5y = e^{-3x}$ corresponding to e^{-3x} as $(1/32) e^{-3x}$.
Now, we will find the particular solution of the equation $y'' - 6y' + 5y = \cos 2x$

$$y_p = \frac{cos2x}{D^2 - 6D + 5} = \frac{cos2x}{-2^2 - 6D + 5} = \frac{cos2x}{1 - 6D} = \frac{(1 + 6D)cos2x}{1 - 36D^2} = \frac{cos2x - 12sin2x}{145}$$

Another method for finding particular solution
Assume the particular solution of the form $Y_p = A \cos 2x + B \sin 2x$. Differentiating this, we have $Y_p' = -2A \sin 2x + 2B \cos 2x$ and $Y_p'' = -4A \cos 2x - 4B \sin 2x$. Substituting these values in the differential equation $y'' - 6y' + 5y = \cos 2x$, we have
$-4A \cos 2x - 4B \sin 2x - 6(-2A \sin 2x + 2B \cos 2x) + 5(A \cos 2x + B \sin 2x) = \cos 2x$
$\Rightarrow -4A \cos 2x - 4B \sin 2x + 12A \sin 2x - 12B \cos 2x + 5A \cos 2x + 5B \sin 2x = \cos 2x$
$\Rightarrow (A - 12B) \cos 2x + (B + 12A) \sin 2x = \cos 2x$
$\Rightarrow A - 12B = 1$ and $B + 12A = 0$
$\Rightarrow A = 1/145$ and $B = -12/145$
$\Rightarrow Y_p = (1/145) \cos 2x - (12/145) \sin 2x$
Now, taking the sum of both particular solutions, the final particular solution of the given second order differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$ is $y_p = (1/32) e^{-3x} + (1/145) \cos 2x - (12/145) \sin 2x$.
Therefore, the complete solution of the differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$
is $y = c_1 e^x + c_2 e^{5x} + (1/32) e^{-3x} +(1/145) \cos 2x - (12/145) \sin 2x$

Fundamental Solution

If y_1 and y_2 are two solutions of the differential equation $y'' + a_1(t)\, y' + a_o(t)\, y = 0$, then y_1 and y_2 are called fundamental solution if and only if y_1 and y_2 are linearly independent, that is,
$W_{y1y2} \neq 0$.

General Solution

If y_1 and y_2 are two fundamental solution of the differential equation $y'' + a_1(t)\, y' + a_o(t)\, y = 0$, and c_1 and c_2 be any two arbitrary constants, then $y(t) = c_1\, y_1(t) + c_2\, y_2(t)$ is said to be the general solution of the given differential equation

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