Bayesian Approach: Chen Distribution Analysis for Record Values Through MCMC Techniques

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Abstract: The Bayes estimate of the Chen distribution's unknown parameters when the data is upper in record values. We present the Bayes estimators and assume Jeffrey's priors on the unknown parameters based on the squared error loss function. Bayes estimators can be generated using numerical integration, but they cannot be obtained explicitly. As a result, we generated a posterior sample using the MCMC method. The evaluation of the results is summarized through a real data.

Keywords: Chen distribution, Record Values, Bayesian Estimation, Simulation and MCMC Techniques.

1. INTRODUCTION

The concept of two-parameter lifetime distributions with a bathtub shape or increasing failure rate function has its origins in the field of reliability engineering and survival analysis. These distributions were developed to model and analyze the reliability and failure patterns of various systems and components over time. And this is given by Chen so it is also called chen distribution. Let $x_i,\ i \geq 1$ be a sequence of independently and identically distributed (i.i.d.) continuous random variables with cdf and pdf. An observation x_j will be termed an upper record value if its value surpasses that of each former observation. Thus x_j, is upper record if $x_j > x_j$, for every $i < j$.

The pdf and cdf of Chen distribution are of the form:

$$f(x;\alpha\beta) = \alpha\beta x^{\beta-1} exp[\alpha\left(1-e^{x^{\beta}}\right)+x^{\beta}], \quad \#(1.1)$$

$$x>0;\ \alpha,\beta>0$$

And cumulative distribution function

$$F(x;\alpha\beta) = 1-exp\left[\alpha\left(1-e^{x^{\beta}}\right)\right], \qquad \#(1.2)$$

$$x>0;\ \alpha,\beta>0$$

Where α and β are unknown parameters.

2. PARAMETER ESTIMATION

Suppose that $\underline{x} = x_{u(1)}, x_{u(2)}, x_{u(3)}, \dots,\ x_{u(n)}$ be first upper records of size n from Chen distribution from pdf $f(x;\alpha,\beta)$ and cdf $F(x;\alpha,\beta)$. Then the joint distribution of the first n upper record values is given by AHSANULLAH (1995) as shown below:

$$f(x;\alpha,\beta) = f\left(x_{u(n)},\alpha,\beta\right)\ \prod_{i=1}^{n-1}\frac{f\left(x_{u(i)}\right)}{1-F\left(x_{u(i)}\right)}, \quad (2.1)$$

2.1. MAXIMUM LIKELIHOOD ESTIMATION

Suppose we observe the first n upper record values $x = x_{u(1)}, x_{u(2)}, x_{u(3)}, \dots,\ x_{u(n)}$ from $Chen(\alpha,\beta)$ distribution with pdf and cdf given by equation (1.1) and (1.2) respectively. Then from (1.1), (1.2) and (2.1), and then the likelihood first n upper record values is given by:

$$\mathrm{L}\left(\alpha,\beta|\underline{x}\right) = \alpha^{n}\beta^{n}\prod_{i=1}^{n} e^{x_{u(i)}^{\beta-1}}$$

$$\prod_{i=1}^{n}\exp\left(e^{x_{u(i)}^{\beta}}\right)\exp\left[\alpha\left(1-e^{x_{u(n)}^{\beta}}\right)\right] \quad \#(2.1.1)$$

Now log-likelihood functions is given by:

$$l\left(\alpha,\beta|\underline{x}\right) = logL\left(\alpha,\beta|\underline{x}\right)$$

$$l\left(\alpha,\beta|\underline{x}\right) = nlog\alpha + nlog\,\beta + (\beta - 1)\sum_{i=1}^{n} logx_{u(i)}n + \sum_{i=1}^{n} x_{u(i)}^{\beta} + \alpha(1 - e^{x_{u(n)}^{\beta}})$$

$$l\left(\alpha,\beta|\underline{x}\right) = nlog\alpha + nlog\,\beta + (\beta - 1)\sum_{i=1}^{n} logx_{u(i)}n + \sum_{i=1}^{n} x_{u(i)}^{\beta} + \alpha - \alpha\, e^{x_{u(n)}^{\beta}} \quad \#(2.1.2)$$

Taking derivative of equation (2.1.1.) and equating them to zero, we obtain the normal equation for α and β

$$\frac{\partial}{\partial\alpha}\left[l\left(\alpha,\beta|\underline{x}\right)\right] = \frac{n}{\alpha} + 1 - e^{x_{u(n)}^{\beta}} = 0 \quad \#(2.1.3)$$

And

$$\frac{\partial}{\partial\beta}\left[l(\alpha,\beta|\underline{x})\right] = \frac{n}{\beta} + \sum_{i=1}^{n}\log x_{u(i)} - \alpha e^{x_{u(n)}^{\beta}} x_{u(n)}^{\beta}\log(x_{u(n)}) + \sum_{i=1}^{n} x_{u(i)}^{\beta}\log(x_{u(i)}) = 0 \quad \#(2.1.4)$$

Equation (2.1.3) yields MLE of α to be

$$\hat{\alpha} = \frac{n}{(1-e^{x_{u(n)}})} \quad \#(2.1.5)$$

Substituting equation (2.1.5) into equation (2.1.4), the MLE of β may be obtained through solving nonlinear equation

$$\frac{n}{\beta} + \sum_{i=1}^{n}\log x_{u(i)} - \alpha e^{x_{u(n)}^{\beta}} x_{u(n)}^{\beta}\log(x_{u(n)}) + \sum_{i=1}^{n} x_{u(i)}^{\beta}\log(x_{u(i)}) = 0 \quad \#(2.1.6)$$

Here should be notes that the equation (2.1.6) is complicated to solve mathematically because this is a nonlinear equation, so we may use ***R*** software to solve this equation and find MLE of the unknown parameters along with required class intervals.

2.2. BAYES ESTIMATION USING MCMC

The first step in Bayesian estimation is to formulate prior distribution. Assuming a joint prior distribution of the parameter α and β in the form.

$$\pi(\alpha,\beta) = \pi_1(\beta|\alpha)\pi_2(\alpha|\beta) \quad \#(2.2.1)$$

Where

$$\pi_1(\beta|\alpha) \propto \frac{1}{\beta} \quad \beta > 0 \quad \#(2.2.2)$$

Which is Jeffrey's non-informative prior distribution (see JAFFREY (1961)) the parameter β for fixed value of a parameter α

$$\pi_2(\alpha|\beta) \propto \frac{1}{\alpha} \quad \alpha > 0 \quad \#(2.2.3)$$

Now putting values from equation (2.2.1) and (2.2.2) in equation (2.2.3) so joint prior will be,

$$\pi(\alpha,\beta) = \frac{1}{\beta} * \frac{1}{\alpha} \quad \#(2.2.4)$$

The joint posterior density of α and β is

$$\pi^*(\alpha,\beta|\underline{x}) = \frac{L(\alpha,\beta|\underline{x})\,\pi(\alpha,\beta)}{\int_0^\infty\int_0^\infty L(\alpha,\beta|\underline{x})\,\pi(\alpha,\beta)d\alpha d\beta} \quad \#(2.2.5)$$

Now, we utilise MCMC technique to produce samples from the posterior distribution and then compute Bayes Estimator of g(α, β) under the squared error loss(SEL) function. The ratio of two integrals provided by (2.2.5) cannot be achieved in a closed form, therefore we use MCMC methods for approximation. See for example ROBERT (2010)

2.2.1. MCMC APPROACH

By multiplying the likelihood by the joint prior, the equation for joint posterior up to proportionality may be desired as,

$$Posterior = Likelihood * Prior$$

$$\pi(\alpha,\beta|a,b) = \alpha^{n-1}\beta^{n-1}\prod_{i=1}^{n}[x_{u(i)}]^{\beta-1}\exp\left[\alpha(1-e^{x_{u(n)}^{\beta}})\right]\prod_{i=1}^{n}e^{x_{u(i)}^{\beta}} \quad \#(2.2.1.1)$$

2.3. EMPIRICAL ANALYSIS

In this chapter, we analyse real data of LAWLESS (2011). It shows the number of 1000s of cycles to failure for electrical appliances in a life test. The data set contains 60 observations, ranging from 0.01 to 9.70. The data set can be used to analyse the reliability of the electrical appliances.

0.01, 0.03, 0.05, 0.06, 0.06, 0.08, 0.12, 0.14, 0.16, 0.21, 0.38, 0.46, 0.47, 0.55, 0.57, 0.83, 0.91, 0.96, 0.99, 1.06, 1.08, 1.09, 1.17, 1.27, 1.27, 1.35, 1.39, 1.47, 1.57, 1.64, 1.70, 1.89, 1.93, 2.00, 2.16, 2.29, 2.32, 2.33, 2.62, 2.78, 2.81, 2.88, 2.99, 3.12, 3.24, 3.71, 3.79, 3.85, 3.91, 4.1, 4.10, 4.11, 4.31, 4.51, 4.58, 5.26, 5.29, 5.58, 6.06, 9.70

Table 2.1 ESTIMATES OF α AND β OBTAINED BY MLE AND MCMC

Method	Parame-ter	Point	Interval	Leng-th
MLE's	α β	1.52 0.50	[1.02 2.02] [0.44, 0.55]	1.00 0.11
Bayes Estimates	α β	1.25 0.24	[2.14, 2.86] [-1.62, -1.19]	0.72 0.43

3.CONCLUSION

From the results obtained in table 2.1 are based on real data. It is seen that the effectiveness of Bayes estimators to non-informative Jeffery's prior is quite close to that of the MLEs.

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