© July 2024| IJIRT | Volume 11 Issue 2 | ISSN: 2349-6002

Study of Some Fixed-Point Theorems in Partial Cone Metric Space for Contractive Mapping

[1] Anil Kumar Yadav,[2]Dr. Jawed Ahmad, [3]Dr. Prabhat Kumar Singh

[1] Research Scholar, University Department of Mathematics, Kolhan University, Chaibasa, Jharkhand, India

[2]Associate professor, Department of Mathematics, Jamshedpur Women's Univesity Jamshedpur, Jharkhand India

[3]Assistant Professor, Department of Statistics Jamshedpur co-operative college Jamshedpur, Jharkhand, India

Abstract- Fixed point theorem is the developments of new creative methods to use the fundamental tools of the functional analysis. Functional analysis embodies the abstract approach in analysis. It gives many fundamental notions relevant for the description, analysis, numerical approximation, medicine ecology, agroindustries and computer simulation process. It also discovers solutions to problems occurring in pure, applied and social sciences.

Keywords: Fixed Point, partial cone metric space, mapping, contractive mapping, Topological spaces, numerical approximation

INTRODUCTION

The recent development in science and technology the study of various Mathematical branches. Fixed point theorem has fascinated hundreds of researchers since 1922 with the celebrated Banach's Fixed Point Theorem. There exists a very vast literature on the topic and this is a very active field of research at present.

A self map T of a metric space X is said to have fixed point x if Tx=x. theorems concerning the existence and properties of fixed points are known as Fixed Point Theorems. It provides a powerful tool to discover solutions to problems occurring in Pure, applied and social sciences for engineering, medicine, agro-industries, Economics, Bio-economics and space science etc.

The first result of fixed point was presented by French mathematician H.Poincare (1985) while Studying the vector distribution as a map of the surface at the point . He translated a point as a vector based distribution to which an index was assigned. These singularities are the fixed points. Since then, a vast amount of work has been done in this field; most noteworthy among them are Banach Fixed Point Theorem. Schauder Fixed Point theorem and Brouwer's Fixed Point Theorem.

Fixed point theorems give the condition under which maps have solutions. The theory itself is a beautiful mixture of analysis, topology and geometry. Over the last years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of non –linear phenomena. However, for many practical situations the conditions that have to be imposed in order to guarantee the existence of fixed point are too strong. Considerable amount of researchers have been done on fixed point theorem for various type mapping in the last few years.

Fixed Point Theory Play's a major role in applications of many branches of mathematics. In 1922, police mathematician Banish proved a very important result regarding a contraction mapping, known as the Banach certain principle (1992)

In 1984, wang et al., introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces In 1980, Rzepecki (1980) introduced a generalized metric d_E on a set X in a way that d_E: X×X → P, replacing the set of real numbers with a Banach space E in the metric function where P is a normal cone in E with a partial order ≤. Seven years later, Lin (1987) considered the notion of cone metric space by replacing real numbers with a cone P in the metric function in which it is called a k-metric. On the otherhand, L-g. Huang and X Zhang (2007) announced the notion of a cone metric space by replacing real numbers with an ordering Banach space to define cone metric space which is the same as either the definition of Rzepecki or of Lin . In the same paper ,they proved some important fixed point Theorem including Banach's Fixed Point Theorem in Cone Metric Space , After that cone metric spaces have been studied by many other authors (see [(2007),(2008),(2009)].

In 1994, Mathews (1994) , Introduced the notion of partial metric space as a part of the study of mentioned semantics of data flow networks related to in computer science.

The concept of partial cone metric space play a very important role not only in Topology but also in other branches of science involving mathematics especially in computer sciences information science and biological science. He generalized the concept of metric space in the sense that the distance from point to itself need not be equal to zero.

Very recently in 2013, has introduced the concept of the partial cone metric space. In cone metric spaces. The self –distance for any point need not equal zero. specially from the point of sequence, a convergent sequence need not have unique limit, A cone metric space is Hausdorff and so has the property that any singleton is a closed subset of the space. In [1] authors have established a new fixed-point theorem for contractive type mappings in the setting and partial cone metric spaces. In Sequel D. Dey & M. Saha (2013) proved some fixed point theorems in partial cone metric spaces .

PRELIMINARIES NOTES

First , we invite some standard notations and definitions of Partial metric Space , Cone metric Spaces and Partial Cone Metric Spaces and some Properties as follows :

Definition 2.1 [Partial Metric Spaces]

Partial metric space on a non empty set X is a function P: X x X $\rightarrow R^+$ Such that for all x , y , z ∈ X :

(P_1) : $0 \leq P(x , y) \leq p(x,y)$;

(p_2): $x = y \Leftrightarrow p(x,x) = p(x, y) = p(y,y)$;

(p_3) : $p(x,y) = p(y,x)$

(p_4): $p(x, y)) \leq p(x,z) + p(z, y) + p(z ,z)$

A Partial Metric space is a Pair (X, p) such that X is a non –empty set and p is a Partial Metric on X it is clear that if p(x,y)=o , Then from (p_1) &(p_2) , we obtained x=y . But if x=y , p(x ,y) may not be zero .

Definition 2.2

Let E be a real Banach Space on P a subset of P is called if and only if ;

(i) P is closed , non empty w $p \neq \{0\}$;

(ii) $ax + by \in p$, for all $x, y \in p$ w non negative real numbers a; b ;

(iii) $x \in p$ w- $x\in p \Rightarrow x=0 \Leftrightarrow p\cap(-p) = \{0\}$.

Given a cone P⊂E , a partial ordering is defined as≤ on E with respect to P by $x \leq y$ iff $y - x \in p$. It is denoted as $x \leq y$ with stand y-x Intp where Intp denotes the interior of p . The cone P is called normal if if there is a number k>0 such that for all x , y ∈ E,

$$0\leq x \leq y \implies \|x\| \leq 1 < \|y\|$$

The least positive number k satisfying above is called normal constant of p .The cone p is called regular if every increasing sequence which is is bounded from above is convergent .That is ,if (x_n) is sequence such that $x_1\leq x_2\leq \ldots\ldots\ldots\leq x_n\leq \ldots\leq y$ for some y ∈ E, the there is x ∈ E such that $\|x_n$- x $\rightarrow 0$ $(\rightarrow \infty)$.Equivalently , the cone p is regular iff every decreasing sequence which is bounded from below is convergent . It is well known that a regular cone is normal cone .

Definition 2.3 (6) : Let X be a non empty set .Suppose wd: X x X → E a mapping such that

(i) $0 \leq d(x,y)$, for all x , y ∈ X and d(x,y) = 0if and only if x= y ;

(ii) d (x ,y) = d(y,x)) , for all x , y ∈ X

(ii) $0 \leq d(x,y)+d(x ,z) +d (z ,y)$, for all x , y ,z ∈ X

Then d is called a cone metric on X and (X , d) is called a cone metric space .

Motivated by this heatful generalization of metric space , Recently , Sonmez , A [13] Introduced the notion of Partial Cone Metric Space X x X $\rightarrow R^+$ Such that for all x , y , z ∈ X :

(P_1) : $0 \leq P(x , y) \leq p(x,y)$;

(p_2): $x = y \Leftrightarrow p(x,x) = p(x, y) = p(y,y)$;

(p_3) : $p(x,y) = p(y,x)$

(p_4): $p(x, y)) \leq p(x,z) + p(z, y) + p(z ,z)$

and its topological characterization . He has also developed some important fixed Point Theorems in this generalized setting.

We now state the following definition of Partial Cone Metric Space due to Sonmez , A [13].

Definition 2 & [Partial Cone Metric Space [13].] ; . A Partial Cone Metric Space on a non empty set X is a function P : X x X $\rightarrow R^+$ Such that for all x , y , z ∈ X :

(PCM_1) : $\theta \leq P(x, y) \leq p(x,y)$;

(PCM_2): $x = y \Leftrightarrow p(x,x) = p(x, y) = p(y,y)$;

(PCM_3) : $p(x,y) = p(y,x)$

(PCM_4): $p(x, y)\,) \leq p(x,z) + p(z, y) + p(z ,z)$

A Partial Cone Metric Space is a pair (X , p) such that X is a non empty set and P is partial cone metric space on X .

It is clear that , if $P(x,y) = \theta$,thus from (PCM_1) & (PCM_2) , x =y . But if x=y , P(x ,y) may not be equal to θ .

A Cone Metric Space is a Partial e Metric Space , but ∃ Partial Cone Metric Spaces which are not Cone Metric Space . Since Cone Metric Space is a also Topological Spaces , a Cone Metric Space is a Partial Cone Metric Space so , any Partial Cone Metric Space is a also Topological Space .

<u>Example</u> : Let $E = R^2$

$P = \{(x ,y) \in E : x , y \geq 0 \}$ and $X = R^+$ and $P : X \times X \rightarrow E$ defined by ρ (x ,y) = (max {x , y} ×max{x , y}) where $\alpha \geq 0$ is a constant .

Then (X , P) is a Partial Cone Metric Space which is not a cone metric space .

Example : - Let $E = l_1$

$P = \{ \{ x_n\} \in l_1 : x_n \geq 0 \}$

Also let $X = \{ (x_n) \in (R+)^w , \sum x_n < \infty \}$

Where $(R+)^w$ be the set of all infinite sequence over R^+ and $P : X \times X \rightarrow E$ defined by

$P (x , y) = (x_1 \vee y_1 , x_2 \vee y_2 , \ldots , , x_n \vee y_n)$

Where the symbol V denotes the max i . e . x v y = Max (x, y) ,then (X, P) is a Partial Cone Metric Space which is not a cone metric space .In the following sequence (X,P) will denote a Partial Metric Space .

We know invite the following definitions and theorems due to Sonmez [13] .

Theorem 1 [12] : Cone Metric Space (X ,P) is a Topological space ,Theorem 2.7 (Theorem 2, (13)] :Let (X , P) be a Partial Cone Metric Space and P be a normal cone with normal constant K , then (X , P) is T_0 .

Theorem 2.8 (Theorem 3, [13]] :- Let (X , P) be a Partial Cone Metric Space and P be a normal cone with normal constant K . Then $\{x_n\}$ convergent to x iff $p(x_n ,x) \rightarrow p(x ,x)$ $(n \rightarrow \infty)$.

Definition 2.9 [13]: - Let (X , P) be a Partial Cone Metric Space , Let $\{ x_n\}$ be a sequence in X and x∈X if for every c ∈IntP ,there is N such that for all n > N , $p (x_n , x) \leq c + P (x , x)$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x , and x is the limit of $\{x_n\}$, we denoted this by

$$\lim_{n\rightarrow\infty} x_n = x \text{ or } x_n \rightarrow x \text{ as } n\rightarrow \infty.$$

(ii)Let (X , P) be a Partial Cone Metric Space , $\{ x_n\}$ be a sequence in X . Then $\{x_n\}$ is Cauchy sequence if there is a ∈ p such that for every ∈>0 there is N such that for all n , m > N

$$\|p(x , x) - a\| < \in .$$

(iii) A Partial Cone Metric Space (X ,P) is said to be complete if every Cauchy sequence in (X , P) is convergent in (X , P) .

Lemma 2.10 [13] :- Let $\{ x_n\}$ be a sequence in Partial Cone Metric Space (X , P) . If a point x is the limit of $\{ x_n\} \cup P (x , y) = P(y , x)$ .Then y is the limit point of $\{x_n\}$.

MAIN RESULTS

In this section, we explain the following fixed point theorem for a self mapping satisfying contractive condition in Partial Cone Metric Spaces.

<u>Theorem 3.1:-</u> Let (X, P) be a complete Partial Cone Metric Space P is a normal cone with constant K supposes the mapping T: X →X satisfies the contractive condition

$P (T_x ,T_y) \leq a_1 p (x ,y) + a_2 P(x, T_x) + a_3 P(y,T_y) + a_4(y , T_x) + a_5(x , T_y)$

Where a_1 , a_2 ,a_3 , a_4 , a_5 so are constant and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T has a unique fixed point in X , and for any x ∈ X , the iterative sequence $(T^n x)$ converges to fixed point

Proof :- Choose $x_0 \in X$ set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$ $\ldots x_{n+1} = Tx_n = T^{n+1}x_0$. Then we have

$P(x_{n+1},x_n) = P(Tx_n , Tx_{n-1})$

$\leq a_1 p (x_n ,x_{n-1}) + a_2 P(x_n, Tx_n) + a_3 P(x_{n-1},Tx_{n-1}) + a_4(x_{n-1} , Tx_n) + a_5(x_n , Tx_{n-1})$

$\leq a_1 p (x_n ,x_{n-1}) + a_2 P(x_n, Tx_{n+1}) + a_3 P(x_{n-1},x_n) + a_4(x_{n-1} , x_{n+1}) + a_5(x_n , x_n)$

$\leq a_1 p (x_n ,x_{n-1}) + a_2 P(x_n, Tx_{n+1}) + a_3 P(x_{n-1},x_n) + a_4[(x_{n-1} , x_n) + (x_{n+1} , x_n)]$

$[1-(a_2+a_4)]\, P(x_{n+1} , x_n) \leq (a_1 + a_3 + a_4)\, P(x_{n-1} ,x_n)$

$\Rightarrow P(x_{n+1} , x_n) \leq (a_1 + a_3 + a_4)\frac{1}{1-(a2+a4)} P(x_{n-1} ,x_n)$

$\Rightarrow$ $P(x_{n+1}, x_n) \leq K\, p(x_n, x_{n-1})$

Where $\frac{(a1+a3+a4)}{1-(a2+a4)} < 1$

$\therefore$ $P(x_{n+1}, x_n) \leq K^n\, p(x_1, x_0)$

$\forall$ n , m > k we have

$P(x_m,x_n) \leq p(x_m, x_{m-1}) + P(x_{m-1}, x_{m-2}) + \ldots P(x_{n+1},x_n) - \sum_{k=1}^{m-n-1} p(x_{m-k},x_{m-k})$

$\leq p(x_m, x_{m-1}) + P(x_{m-1}, x_{m-2}) + \ldots P(x_{n+1},x_n)$

$\leq (k^{m-1}+k^{m-2}+\ldots..k^n)p(x_1, x_0)$

$\leq \frac{k}{1-k}\, p(x, x_0)$

Since p is normal

There $\|p(Xm, Xn)\| \leq k\, \frac{k}{1-k} \|p(X1, X0)\| \to 0$ as $n \to \infty$,

i.e. $\|p(Xm, Xn)\| \to 0$ as m ,n $\to \infty \Rightarrow p(x_m, x_n) \to 0$ as m , n $\to \infty$,

Hence $\{x_n\}$ is a Chauchy Sequence by completeness of X , there exist $u \in X$ such that $x_n \to u$ ($n \to \infty$). i.e $\lim_{n\to\infty} p(x_n,x) = p(x,y) = \lim_{n\to\infty} p(x_n,x_n) = 0$

Now we have

$P(Tu_0,u) \leq p(Tu, Tx_n) + P(u, Tx_n) - P(Tx_n, Tx_n)$

$\leq a_1\, p(u_0, x_n) + a_2 P(u_0, Tu) + a_3 P(x, Tx_n) + a_4 P(x_n, Tx_n) + a_5 P(u, Tx_n)$

$\leq a_1\, p(u, x_n) + a_2 P(Tu, u) + a_3 P(x_n, x_{n+1}) + a_4 P(Tu, x_n) + a_5 P(u, x_{n+1}) + P(x_{n+1},u)$

$\leq a_1\, p(u,u) + a_2 P(Tu, u) + a_3 P(u, u) + a_4 P(u, Tu) + a_5 P(u, u) + d(x_{n+1},u)$

So , using the condition at normality of con$\|P(Tu, u)\|$e

$\|P(Tu, u)\| \leq a_1\|P(u,u)\| + a_2\|P(Tu, u)\| + a_3\|P(u, u)\| + a_4 \|P(u, Tu)\| + a_5\|P(u, u)\| + \|P(x, u)\| \to 0$

$\leq (a_2+a_4)\|P(Tu,u)\| \to$

$\Rightarrow \|P(Tu,u)\| = 0$; which implies that Tu= u . so u is the fixed point of T .

Let V be another fixed Point of T in X then

$P(u, v) = P(Tu,Tv)$

$\leq a_1\, p(u, v) + a_2 P(u, Tu) + a_3 P(v, Tv) + a_4 P(v, Tv) + a_5 P(u, v)$

$\leq a_1\, p(u, v) + a_2 P(u, u) + a_3 P(v, v) + a_4 P(v, u) + a_5 P(u, v)$

$= (a_1+2a_4)P(u, v)$.

Since P is normal cone with normal constant k

$\|P(u, v)\| \leq 0$.

$\therefore$ u = v

Therefore fixed point is equal .

CONCLUSION

In section 3.3, we established and improved the results of Dey and Saha (2012).

REFERENCE

[1] Abbas , M and Jungck , G . (2008) , Common fixed point results for non commuting mapping without continuity in cone metric spaces , J . Math . Anal . Appl. 341 , 416-420.

[2] Abhas , M & Rhoades , B . E . (2009) , Fixed and Periodic Point results in cone metric spaces . Applied Mathematics Letters 22 511-515.

[3] Azam, A. and Arshad, M. (2008), Kannan Fixed Point Theorem on Generalize Metric Spaces, J. Non linear. Sci. Appl., 1(1): 45-48.

[4] Banach ,S; (1922) ,Sur Lis operations dans les ensembles abstraits etleur application and equations integrals . Fund Math ; 3 , 133-181.

[5] Beiranvand, A., Moradi, S., Omid, M. and Pazandeh, H. (2009), Two Fixed Point Theorem for special mapping, arXiv: 0903. 1504v1 [Math. FA].

[6] Branciari, A.(2000), A Fixed Point Theorem of Banach- Caccippoli Type on a Class of Generalized Metric Spaces, Publ. Math. Debrecen, 57., 31-37.

[7] Brouwer, L., (1912), Uber Abbildungen Von Manning Faltigkeiten, Math Ann.70: 97- 115.

[8] Browder, F.E.(1968), On the convergence of successive approximations for non- linear functional equations, Indag. Math. 30, 27-35.

[9] Chen, J. and Li, Z. (2007), Common Fixed Point for Banach Operator 1466-1475.

[10] Chen, J. and Li, Z. (2007), Common Fixed Point for Banach Operator Pairs in Best Approximation, J. Math. Anal. Appl. 336: 1466-1475 .theory Appl. 2012 ,13.

[11] Chu, S.C. and Diaz, J.B(1965), Remarks on a generalization of Banach's principle of contraction mappings, Jour. Math. Anal. Appl., 11, 440-446.

[12] Deimling, K.(1985) Non Linear Functional Analysis, Springer –Verlag.

[13] Dubey, L.S.(1972), On common fixed points, Riv.Math. Univ. Parma, 3(1), 119-124.

[14] Huang, L.-G. and Zhang, X., (2007), Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332(2), 1468-1476.

[15] Kannan, R., (1968), Some results on fixed points, Bull. Calcutta Math. Soc., 60, 71-76.

[16] K¨unzi, H.P.A., (Dordrecht, 2001), Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology, Handbook of the History of General Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., 3, 853-968.
[17] Lin, S.D., (1987), A common fixed point theorem in abstract spaces, Indian Journal of Pure and Applied Mathematics, 18(8), 685-690.
[18] Matthews, S., (1994), Partial Metric Topology, Proceedings of the 8th Summer Conference on Topology and its Applications, Annals of The New york Academy of Sciences, 728, 183-197.