Darcy Brinkmann Rayleigh Benard Triple Diffusive Convection in A Composite System with Temperature Gradients and Diffusion Thermo Effect: Rigid Free Boundaries Without Surface Tension

SHIVARAJA. J. M[1], SUMITHRA. R.[2], RASHI. S. K.[3], RAMESHA. S. N.[4], SOWMYA. K.[5]

[1]*Assistant Professor, Department of Mathematics, Bengaluru North University, Tamaka, Kolar, Karnataka*

[2]*Professor, Government Science College (Autonomous), Nrupathunga University, Nrupathunga Road, Bengaluru, Karnataka*

[3, 4, 5]*Post Graduate Students, Department of Mathematics, Bengaluru North University, Tamaka, Kolar, Karnataka*

Abstract— The influence of Diffusion – Thermo (Dufour) and Non-Darcian effect on Three - Component Rayleigh Benards (TCRB) convection in a two-layered system, has investigated analytically by using the Regular Perturbation technique. The upper and lower boundaries of the composite system are rigid and rigid-free with surface tension respectively, these boundaries are insulating to both heat and mass. At the interface of the composite system, the heat and heat flux, normal velocity, normal and shear stress, and mass and mass flux, are assumed to be continuous. The Non-Darcy term and Dufour term are employed in the momentum and energy equation respectively. The impact of various physical parameters on TCRB convection is investigated in a detailed manner and results are portrayed graphically.

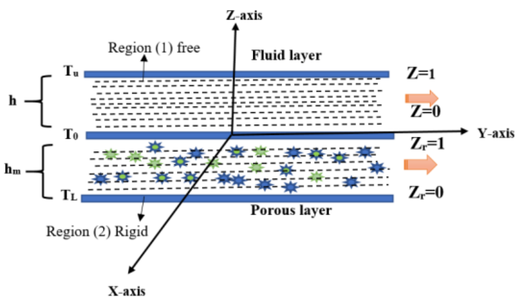
Index Terms- Rayleigh Benard Convection, Dufour effect, Two – Component Convection, Composite layer.

I. INTRODUCTION

Variations in concentration cause the Dufour (Diffusion Thermo) effect, which is the flow of energy. It is the combined impact of irreversible processes that causes energy to flow due to a concentration gradient. The survey of literature focuses primarily on the various classes of techniques related to our field of study, which is the effects of Dufour and non-Darcian effects on three-component Rayleigh Benard (TCRB) convection in a two-layered system with rigid-free boundaries. Research has been done on the issues of triple diffusive/three-component convection in fluids by Pearlstein et al. [7] and Lopez et al. [1]. Rudraiah and Vortmeyer [9] used a gravitationally stable density gradient to examine the linear stability of the system. They did this by considering the porous medium. Poulikakos [8] stated that adding a third diffusive component with low diffusivity can significantly alter the character of the convective instabilities in the system. The compound/composite layer has limited impact on triple diffusional convection. A more advanced technique to investigate triple diffusive (three-component) Marangoni convection in a two-layer/composite layer has been devised by Sumithra R [14]. It obtains an algebraic equation for the Marangoni number, solves the resulting eigenvalue issue exactly, and thoroughly examines the effects of changing various physical parameters on it. Manjunatha [4] presented a different approach that examines the impact of temperature profiles on triple diffusive convection. On the other hand, Mokhtar et al. [2] looked at how internal heat generation affects a two-layer system or a composite system with Marangoni effects. They use the well-known method of normal mode analysis to convert the system of PDEs into a system of ODEs. The regular perturbation methodology effectively solves the ODE system, which is employed to handle other Rayleigh Benard (RB) convection-related issues. The influences of non-Darcian effects and thermal diffusion (Soret) effects on Rayleigh Benard (RB) convection in a two-layer composite layer were investigated in a conceptually equivalent study using a similar approach proposed by Sumithra et al. [13]. We assume that both borders of this system are rigid. The regular perturbation approach, a widely used method, solves Rayleigh Benard's (RB) convection with the Darcy Brinkmann model. The Rayleigh number can be derived algebraically by solving the linked equations. Rayleigh Benard (RB) convection has been studied by Park et al. [4] with the assumption that the heat flux is constant at the boundary.

II. MATHEMATICAL FORMULATION

The physical model under the consideration of a composite layer system with a horizontal two–component fluid-saturated, incompressible, isotropic, and sparsely packed porous layer is of thickness $d_{\alpha r}$ and a fluid layer of thickness d_{α}. The boundaries of the composite layer system are considered rigid-free; these boundaries act like heat and mass insulators along with maintained, distinct concentration and temperature.



The origin point of the Cartesian coordinate system is taken exactly at the intersection of the porous and fluid layer along with the direction of the z-axis vertically upwards. In addition to that, for the effect of density variation, the Boussinesq approximation is included. Under these following assumptions, the governing equations are, the continuity, momentum, temperature, concentration, and state equations as follows.

For the region – 1 (Fluid layer)

$$\nabla.\overrightarrow{q_\alpha} = 0 \tag{1}$$

$$\rho_0\left[\frac{\partial\overrightarrow{q_\alpha}}{\partial t_\alpha} + (\overrightarrow{q_\alpha}.\nabla)\overrightarrow{q_\alpha}\right] = -\nabla P_\alpha + \mu_\alpha\nabla^2\overrightarrow{q_\alpha} - \rho_\alpha\, g_\alpha\,\widehat{k} \tag{2}$$

$$\frac{\partial T_\alpha}{\partial t_\alpha} + (\overrightarrow{q_\alpha}.\nabla)T_\alpha = \kappa_1\,\nabla^2 T_\alpha + \kappa_2\,\nabla^2 C_\alpha \tag{3}$$

$$\frac{\partial C_\alpha}{\partial t_\alpha} + (\overrightarrow{q_\alpha}.\nabla)C_\alpha = \kappa_3\,\nabla^2 C_\alpha \tag{4}$$

$$\frac{\partial C_\beta}{\partial t_\alpha} + (\overrightarrow{q_\alpha}.\nabla)C_\beta = \kappa_4\nabla^2 C_\beta \tag{5}$$

$$\rho_\alpha = \rho_0\,[\,1 - \alpha_1\,(T_\alpha - T_i) + \alpha_{C_\alpha}\,(C_\alpha - C_{\alpha i}\,)] + \alpha_{C_\beta}\,(C_\beta - C_{\beta i}\,)] \tag{6}$$

For the region – 2 (Porous layer)

$$\nabla_{\alpha r}.\vec{q}_{\alpha r} = 0 \tag{7}$$

$$\left[\frac{\rho_0}{\phi}\right]\frac{\partial\vec{q}_{\alpha r}}{\partial t_{\alpha r}} = -\nabla_{\alpha r}p_{\alpha r} - \frac{\mu_{\alpha r}}{K_r}\vec{q}_{\alpha r} + \mu_{\alpha r}\nabla^2\vec{q}_{\alpha r} - \rho_{\alpha r}\;g_{\alpha r}\,\hat{k} \tag{8}$$

$$A\left[\frac{\partial T_{\alpha r}}{\partial t_r}\right] + (\vec{q}_{\alpha r}.\nabla_{\alpha r})T_{\alpha r} = \kappa_{1r}\nabla^2_{\alpha r}T_{\alpha r} + \kappa_{2r}\,\nabla^2_{\alpha}C_{\alpha r} \tag{9}$$

$$\phi\left[\frac{\partial C_{\alpha r}}{\partial t_{\alpha r}}\right] + (\vec{q}_{\alpha r}.\nabla_{\alpha r})C_{\alpha r} = \kappa_{3r}\;\nabla^2_{\alpha r}\,C_{\alpha r} \tag{10}$$

$$\phi\left[\frac{\partial C_{\beta r}}{\partial t_{\alpha r}}\right] + (\vec{q}_{\alpha r}.\nabla_{\alpha r})C_{\beta r} = \kappa_{4r}\;\nabla^2_{\alpha r}\,C_{\beta r} \tag{11}$$

$$\rho_{\alpha r} = \rho_0\,[\,1 - \alpha_{1r}\,(T_{\alpha r} - T_{ri}) + \alpha_{C_{\alpha r}}\,(C_{\alpha r} - C_{\alpha ri}\,)] + \alpha_{C_{\beta r}}\,(C_{\beta r} - C_{\beta ri}\,)] \tag{12}$$

Where, $\overrightarrow{q_\alpha} = (u_{\alpha 1}, v_{\alpha 1}, w_{\alpha 1})$ is velocity, t_α is the time, ρ_0 is the referring density of fluid, μ_α is the fluid viscosity, P_α is pressure, ρ_α is the density of fluid, g_α is the gravitation, T_α is the temperature, κ_1 is the thermal diffusivity, κ_3& κ_4 are the solutal diffusivities, κ_2 is the Dufour coefficient, C_α is the concentration, α_1 is the coefficient of thermal expansion, α_{C_α}& α_{C_β} are the solutal analogs of α_1, $\phi, \mu_{\alpha r}$ & K_r are the porosity, effective viscosity and permeability of the porous medium, A is the heat capacities ratio. Subscript "r" refers to physical quantities which are defined in the porous medium.

III. LINEAR STABILITY THEORY

The fluid layer is stable in the composite system by assuming the primary state is inactive, which is quiescent. Therefore there is no fluid motion, so, setting the velocity vector $\vec{q}$ is zero, where, the conduction will happen only by transforming mass and heat assuming the following assumptions, for fluid layer $\overrightarrow{q_\alpha} = [u_{\alpha 1},\ v_{\alpha 1},\ w_{\alpha 1}] = 0$

and $[T_\alpha, C_\alpha, C_\alpha\ P_\alpha,\ \rho_\alpha] = [T_{\alpha b}(z),\ C_{\alpha b}(z), C_{\beta b}(z)\ P_{\alpha b}(z),\ \rho_{\alpha b}(z)]$ for the porous layer, $\vec{q}_{\alpha r} = 0$ and $[T_{\alpha r},\ C_{\alpha r}, C_{\beta r}, P_{\alpha r},\ \rho_{\alpha r}] = [T_{\alpha rb},\ C_{\alpha rb}(z_r), C_{\beta rb}(z_r),\ P_{\alpha rb}(z_r),\ \rho_{\alpha rb}(z_r)]$ finally, we get the temperature and the concentration distributions for fluid and porous layers, after substituting these basic states to the equations (1) to (10).

Temperature distribution and concentration distribution in the fluid layer ($0 \le z \le d_\alpha$)

$$T_{\alpha b}(z) = \frac{(T_\infty - T_i)\, z}{d_\alpha} + T_i \tag{13}$$

$$C_{\alpha b}(z) = \frac{(C_{\alpha\infty} - C_{\alpha i})\, z}{d_\alpha} + C_{\alpha i} \tag{14}$$

$$C_{\beta b}(z) = \frac{(C_{\beta\infty} - C_{\beta i})\, z}{d_\alpha} + C_{\beta i} \tag{15}$$

Temperature distribution and concentration distribution in the porous layer ($-d_{\alpha r} \le z_r \le 0$)

$$T_{\alpha rb}(z_r) = \frac{(T_i - T_{\infty r}) z_r}{d_{\alpha r}} + T_i \tag{16}$$

$$C_{\alpha rb}(z_r) = \frac{(C_{\alpha i} - C_{\alpha r}) z_r}{d_{\alpha r}} + C_{\alpha i} \tag{17}$$

$$C_{\beta rb}(z_r) = \frac{(C_{\beta i} - C_{\beta r}) z_r}{d_{\alpha r}} + C_{\beta i} \tag{18}$$

Temperature distribution and concentration distribution at the interface.

$$T_i = \left[\frac{k_1\, d_{\alpha r}\, T_\infty + d_\alpha\, k_{1r}\, T_{\infty r}}{k_1\, d_{\alpha r} + d_\alpha\, k_{1r}} + \frac{d_{\alpha r} k_2\, C_{\alpha\infty} + k_2}{k_1\, d_{\alpha r} + d_\alpha} - C_{\alpha i}\left(\frac{d_\alpha\, k_{2r} + d_{\alpha r}\, k_2}{k_1\, d_{\alpha r} + d_\alpha\, k_{1r}}\right)\right. \tag{19}$$

$$C_{\alpha i} = \frac{k_3\, C_{\alpha\infty} d_{\alpha r} + d_\alpha\, k_{3r} C_{\alpha r}}{d_\alpha\, k_{3r} + d_{\alpha r}\, k_3} \tag{20}$$

$$C_{\beta i} = \frac{k_4\, C_{\beta\infty} d_{\alpha r} + d_\alpha\, k_{4r} C_{\beta r}}{d_\alpha\, k_{4r} + d_{\alpha r}\, k_4} \tag{21}$$

Now established the perturbed quantities to analyze the linear stability of the primary state solution.

For fluid layer

$$[\overrightarrow{q_\alpha}, P_\alpha, C_\alpha, C_\beta, T_\alpha, \rho_\alpha] = \left\{ \begin{array}{l} [0,\ P_{\alpha b}(z), C_{\alpha b}(z), C_{\beta b}(z) T_{\alpha b}(z), \rho_{\alpha b}(z) \\ \quad + [\overrightarrow{q_\alpha}, P'_\alpha, S'_\alpha, S'_\beta\ \theta'_\alpha, \rho'_\alpha] \end{array} \right. \tag{22}$$

For porous layer

$$[\overrightarrow{q_{\alpha r}}, P_{\alpha r}, C_{\alpha r}, C_{\beta r}, T_{\alpha r}, \rho_{\alpha r}] = \left\{ \begin{array}{l} [0, P_{\alpha rb}(z_r), C_{\alpha rb}(z_r), C_{\beta rb}(z_r), T_{\alpha rb}(z_r), \rho_{\alpha rb}(z_r)] \\ \quad + [\overrightarrow{q_{\alpha r}}, P'_{\alpha r}, S'_{\alpha r}, S'_{\beta r}\ \theta'_{\alpha r}, \rho'_{\alpha r}] \end{array} \right\}$$

(23)

Substitute equations (22) and (23) into equations (1) − (12) after that in order to vanish the pressure term in the perturbed momentum equation by applying curl twice. The resulting perturbed equations are linearized for (1) − (12), introduce non-dimensional quantities, see (Sumithra[]) then Applying normal mode expansions to the perturbed non-dimensionalised quantities using

$$[w_\alpha(z), \theta_\alpha(z), s_\alpha(z), s_\beta(z)]^T = [W_\alpha(z),\ \Theta_\alpha(z),\ S_\alpha(z),\ S_\beta(z)]^T f_\alpha(x,\ y,\ t)$$

$$[w_{\alpha r}(z_r), \theta_{\alpha r}(z_r), s_{\alpha r}(z_r), s_{\beta r}(z_r)]^T = [W_{\alpha r}(z_r), \Theta_{\alpha r}(z_r), S_{\alpha r}(z_r), S_{\beta r}(z_r)]^T f_{\alpha r}(x_r, y_r, t_r)$$

Where, $f_\alpha = f_0\, e^{i(lx+my)-nt}$, $f_{\alpha r} = f_0\, e^{i(l\, x_r + m\, y_r) - n\, t_r}$

Which returns the following ordinary differential equations,

For region – 1(Fluid layer)

$$\left[(D^2 - a^2) + \frac{n}{p_r}\right](D^2 - a^2)\, W_\alpha = R a^2 \theta - R_\alpha a^2 S_\alpha - R_\beta\, a^2\, S_\beta \tag{24}$$

$$(D^2 - a^2 + n)\, \Theta_\alpha + D_\alpha (D^2 - a^2)\, S_\alpha + W_\alpha = 0 \tag{25}$$

$$[\tau_\alpha (D^2 - a^2) + n]\, S_\alpha + W_\alpha = 0 \tag{26}$$

$$[\tau_\beta (D^2 - a^2) + n]\, S_\beta + W_\alpha = 0 \tag{27}$$

For region −2 (porous layer)

$$\begin{bmatrix} \hat{\mu}\beta^2(D_r^2 - a_r^2) \\ +\frac{n_{r\,\beta^2}}{p_{rr}} - 1 \end{bmatrix}(D_r^2 - a_r^2)W_{\alpha r} = R_r {a_r}^2\theta_{\alpha r} - R_{\alpha r}{a_r}^2 S_{\alpha r} - R_{\beta r}a_r^2 S_{\beta r} \quad (28)$$

$$(D_r^2 - a_r^2 + \ n_r\, A)\Theta_{\alpha r} + W_{\alpha r} + D_\beta(D_r^2 - a_r^2)S_{\alpha r} = 0 \quad (29)$$

$$[\tau_{\alpha r}(D_r^2 - a_r^2) + \phi\, n_r\,]\, S_{\alpha r} + W_{\alpha r} = 0 \quad (30)$$

$$\left[\tau_{\beta r}(D_r^2 - a_r^2) + \phi\, n_r\,\right] S_{\beta r} + W_{\alpha r} = 0 \quad (31)$$

Where, for the fluid layer, $\hat{\mu} = \frac{\mu_{\alpha r}}{\mu_{\ \alpha}}$ is the viscosity ratio $D = \frac{d}{dz}$ is the differential operator with respect to $z,\ p_r = \frac{\nu}{\kappa_1}$ is the Prandtl number, $\tau_\alpha = \frac{k_3}{\kappa_1}$ is the ratio of thermal diffusivity to solute1 diffusivity, $\tau_\beta = \frac{k_4}{\kappa_1}$ is the ratio of thermal diffusivity to solute 2 diffusivity $\nu = \frac{\mu_\alpha}{\rho_0}$ is the kinematic viscosity, $R = \frac{g\ \alpha_1(T_i - T_\infty)\ d_\alpha^3}{\nu\ \kappa_\alpha}$ is the Rayleigh number, $R_\alpha = \frac{g_\alpha \alpha_{C_\alpha}(C_{\alpha i} - C_{\alpha\infty})\ d_\alpha^3}{\nu\ \kappa_1}$ is the solute1 Rayleigh number, $R_\beta = \frac{g_\alpha \alpha_{C_\beta}(C_{\beta i} - C_{\beta\infty})\ d_\alpha^3}{\nu\ \kappa_1}$ is the solute2 Rayleigh number and $D_\alpha = \frac{k_2\ (C_{\alpha i} - C_{\alpha\infty})}{\kappa_1\ (T_i - T_\infty)}$ is Dufour coefficient. Similarly for the porous layer the physical quantities are defined with the subscripts "r". For a specific set of numerous parameters,
the principle of change instability is set up to be precisely authentic for fluid or porous layer problems separately and it's far assumed that the precept holds for this situation also, so, fixed $n = n_r = 0$, equations $(24) - (31)$ becomes.
For region – 1 (Fluid layer)

$$(\,D^2 - a^2\,)^2\, W_\alpha = Ra^2\theta - R_\alpha a^2 S_\alpha - R_\beta\, a^2\, S_\beta \quad (32)$$

$$(\,D^2 - a^2\,)\ \Theta_\alpha + W_\alpha + D_\alpha(\,D^2 - a^2\,)S_\alpha = 0 \quad (33)$$

$$[\tau_\alpha(\,D^2 - a^2\,)]S_\alpha + W_\alpha = 0 \quad (34)$$

$$\left[\tau_\beta\ (\,D^2 - a^2\,)\right]S_\beta + W_\alpha = 0 \quad (35)$$

For region -2 (porous layer)

$$[\hat{\mu}\beta^2(D_r^2 - a_r^2) - 1](D_r^2 - a_r^2)W_{\alpha r} = R_r {a_r}^2\theta_{\alpha r} - R_{\alpha r}{a_r}^2 S_{\alpha r} - R_\beta a_r^2 S_{\beta r} \quad (36)$$

$$(D_r^2 - a_r^2)\Theta_{\alpha r} + W_{\alpha r} + D_\beta(D_r^2 - a_r^2)S_{\alpha r} = 0 \quad (37)$$

$$[\tau_{\alpha r}(D_r^2 - a_r^2)]\ S_{\alpha r} + W_{\alpha r} = 0 \quad (38)$$

$$\left[\tau_{\beta r}(D_r^2 - a_r^2)\right] S_{\beta r} + W_r = 0 \quad (39)$$

Finally we obtain the system of ordinary differential equations of order 20, so we need 20 boundary conditions, such boundary conditions are mentioned below.

IV. BOUNDARY CONDITIONS

At the upper boundary, $z = 1$

$W_\alpha(\,z\,) = 0,\quad D^2 W_\alpha(\,z\,) = 0,\quad D\Theta_\alpha\,(\,z\,) = 0,\ DS_\alpha(\,z\,) = 0, DS_\beta(\,z\,) = 0$

At the interface, $z = 0$ and $z_r = 1$

$\hat{T}\,W_\alpha(z) = \ W_{\alpha r}(z_r),\quad \hat{T}\ \hat{d}\,DW_\alpha(z) = D_r W_{\alpha r}(z_r), \Theta_\alpha(z) = \ \hat{T}\,\Theta_{\alpha r}(z_r)$

$D\,\Theta_\alpha(z) = \ D_r\,\Theta_{\alpha r}(z_r),\ S_\alpha(z) = \widehat{S_\alpha}\,S_{\alpha r}(z_r), S_\beta(z) = \ \hat{S}_\beta\,S_{\beta r}(z_r),\quad DS_\alpha(z) = D_r\,S_{\alpha r}(z_r)$

$$\hat{T}\ d^3\beta^2[\,D^3 W_\alpha(z) - 3\,a^2 D\ W_\alpha(z)] = \begin{bmatrix} -D_r W_{\alpha r}(z_r) \\ +\widehat{\mu}\,\beta^2\left[{D_r}^3 W_{\alpha r}(z_r) - 3{a_r}^2 D_r W_{\alpha r}(z_r)\right] \end{bmatrix}$$

At the lower boundary, $z_r = 0$

$W_{\alpha r}(z_r) = 0,\quad D_r W_{\alpha r}(z_r) = 0,\quad D_r\Theta_{\alpha r}\,(z_r) = 0,\ D_r S_{\alpha r}(z_r) = 0, D_r S_{\beta r}(z_r) = 0$

Where,

$\hat{d} = \frac{d_{\alpha r}}{d_\alpha}, \hat{k} = \frac{\hat{d}}{\hat{T}} = \frac{k_{1r}}{k_1}, \hat{T} = \left[\frac{T_\infty - T_i}{T_i - T_{\infty r}}\right], \widehat{S_\alpha} = \left[\frac{C_{\alpha r} - C_{\alpha i}}{C_{\alpha i} - C_{\alpha\infty}}\right], \widehat{S_\beta} = \left[\frac{C_{\beta r} - C_{\beta i}}{C_{\beta i} - C_{\beta\infty}}\right]$

V. REGULAR PERTURBATION METHOD

For the consistent mass and heat fluxes, expanding the physical parameters in terms of horizontal wave number "a" and "a_r"

$$\begin{bmatrix} W_\alpha \\ \Theta_\alpha \\ S_\alpha \\ S_\beta \end{bmatrix} = \sum_{i=0}^{\infty} a^{2i} \begin{bmatrix} W_{\alpha i} \\ \Theta_{\alpha i} \\ S_{\alpha i} \\ S_{\beta i} \end{bmatrix} \ \& \ \begin{bmatrix} W_{\alpha r} \\ \Theta_{\alpha r} \\ S_{\alpha r} \\ S_{\beta r} \end{bmatrix} = \sum_{j=0}^{\infty} a^{2j} \begin{bmatrix} W_{\alpha rj} \\ \Theta_{\alpha rj} \\ S_{\alpha rj} \\ S_{\beta rj} \end{bmatrix} \quad (40)$$

The basic objective of the regular perturbation technique is to substitute equation (40) into equations (32) to (39). The first few terms in a perturbation series expansion are called the perturbation approximations or solutions, the zero-order equations are solved using corresponding zero-order boundary conditions, and solutions are mentioned below.

$$W_{\alpha 0}(z) = 0, W_{\alpha r0}(z_r) = 0, S_{\alpha 0}(z) = \widehat{S_\alpha}, S_{\alpha 0}(z) = \widehat{S_\beta}$$

$$S_{\alpha r0}(z_r) = 1, S_{\beta r0}(z_r) = 1, \Theta_{\alpha 0}(z) = \widehat{T}, \Theta_{\alpha r0}(z_r) = 1$$

The ordinary differential equations at first order are given below.

For region – 1 (Fluid layer)

$$D^4 W_{\alpha 1} - R\widehat{T} + R_\alpha \widehat{S_\alpha} + R_\beta \widehat{S_\beta} = 0 \quad (41)$$

$$D^2 \Theta_{\alpha 1} + D_\alpha (D^2 S_{\alpha 1} - \widehat{S_\alpha}) + W_{\alpha 1} - \widehat{T} = 0 \quad (42)$$

$$\tau_\alpha D^2 S_{\alpha 1} + W_{\alpha 1} - \tau_\alpha \widehat{S_\alpha} = 0 \quad (43)$$

$$\tau_\beta D^2 S_{\beta 1} + W_{\alpha 1} - \tau_\beta \widehat{S_\beta} = 0 \quad (44)$$

For region −2 (porous layer)

$$\widehat{\mu} \beta^2 {D_r}^4 W_{\alpha r1} - {D_r}^2 W_{\alpha r1} - R_r + R_{\alpha r} + R_{\beta r} = 0 \quad (45)$$

$${D_r}^2 \Theta_{\alpha r1} + W_{\alpha r1} + D_\beta {D_r}^2 S_{r\alpha 1} - D_\beta - 1 = 0 \quad (46)$$

$$\tau_{\alpha r} {D_r}^2 S_{\alpha r1} + W_{\alpha r1} - \tau_{\alpha r} = 0 \quad (47)$$

$$\tau_{\beta r} {D_r}^2 S_{\beta r1} + W_{\alpha r1} - \tau_{\beta r} = 0 \quad (48)$$

Boundary conditions corresponding to first order equations in a^2 are as mentioned below.

At the upper boundary, $z = 1$

$$W_{\alpha 1}(z) = 0, \quad DW_{\alpha 1}(z) = 0, \quad D\Theta_{\alpha 1}(z) = 0, \quad DS_{\alpha 1}(z) = 0, DS_{\beta 1}(z) = 0$$

At the interface, $z = 0$ and $z_r = 1$

$$\widehat{T} W_{\alpha 1}(z) = \hat{d}^{\,2} W_{\alpha r1}(z_r), \quad \widehat{T} DW_{\alpha 1}(z) = \hat{d} D_r W_{\alpha r1}(z_r), \quad \Theta_{\alpha 1}(z) = \hat{d}^2 \widehat{T} \Theta_{\alpha r1}(z_r)$$

$$D\Theta_{\alpha 1}(z) = \hat{d}^2 D_r \Theta_{\alpha r1}(z_r), \quad S_{\alpha 1}(z) = \widehat{S_\alpha} \hat{d}^2 S_{\alpha r1}(z_r), \quad S_{\beta 1}(z) = \widehat{S_\beta} \hat{d}^2 S_{\beta r1}(z_r)$$

$$\widehat{T} \hat{d} \beta^2 D^3 W_{\alpha 1}(z) = -D_r W_{\alpha r1}(z_r) + \widehat{\mu} \beta^2 [{D_r}^3 W_{\alpha r1}(z_r)]$$

$$\widehat{T} D^2 W_{\alpha 1}(z) = \widehat{\mu} {D_r}^2 W_{\alpha r1}(z_r), DS_{\alpha 1}(z) = D_r S_{\alpha r1}(z_r), DS_{\beta 1}(z) = D_r S_{\beta r1}(z_r)$$

At the lower boundary, $z_r = 0$

$$W_{\alpha r1}(z_r) = 0, \quad D_r W_{\alpha r1}(z_r) = 0, \quad D_r \Theta_{\alpha r1}(z_r) = 0, \quad D_r S_{\alpha r1}(z_r) = 0, D_r S_{\beta r1}(z_r) = 0$$

The equations (30) and (33) solved using the relevant boundary conditions then we get velocity distributions as below,

$$W_{\alpha 1}(z) = \left[\frac{\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta}{24}\right] (\mathrm{K}_1 + \mathrm{K}_2 z + \mathrm{K}_3 z^2 + \mathrm{K}_4 z^3 + z^4) \quad (49)$$

$$W_{\alpha r1}(z_r) = \begin{bmatrix} \mathrm{K}_5 + \mathrm{K}_6 z_r + \mathrm{K}_7 e^{p z_r} + \mathrm{K}_8 e^{-p z_r} \\ + \frac{(R_{\alpha r} + R_{\beta r} - R_r) {z_r}^2}{2} + \frac{(R_{\alpha r} + R_{\beta r} - R_r)}{P^2} \end{bmatrix} \quad (50)$$

$$\mathrm{K}_1 = \frac{24 \hat{d}^2}{\widehat{T}(\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta) e^{-p}} \begin{Bmatrix} \mathrm{K}_7 (-e^{-p} + 1 - p e^{-p}) + \mathrm{K}_8 e^{-p} (-1 + p + e^{-p}) \\ + \frac{e^{-p} (R_{\alpha r} + R_{\beta r} - R_r)}{2} \end{Bmatrix}$$

$$\mathrm{K}_2 = \frac{24 \hat{d}}{\widehat{T} e^{-p} (\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta)} \begin{Bmatrix} \mathrm{p}\mathrm{K}_7 - p K_7 e^{-p} + \mathrm{K}_8 p e^{-p} (1 - e^{-p}) \\ + (R_{\alpha r} + R_{\beta r} - R_r) e^{-p} \end{Bmatrix}$$

$$\mathrm{K}_3 = \frac{12 \widehat{\mu}}{\widehat{T}(\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta) e^{-p}} [\mathrm{K}_7 P^2 + \mathrm{K}_8 P^2 e^{-2p} + e^{-p} (R_{\alpha r} + R_{\beta r} - R_r)]$$

$$\mathrm{K}_4 = \frac{4}{\hat{d} \beta^2 \widehat{T}(\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta) e^{-p}} \begin{bmatrix} C_7 p (e^{-p} + (p-1)) + C_8 p e^{-2p} (1 - p - e^p) \\ + e^{-p} (R_{\alpha r} + R_{\beta r} - R_r) \end{bmatrix}$$

$$\mathrm{K}_5 = -\mathrm{K}_7 - \mathrm{K}_8 - \frac{(R_{\alpha r} + R_{\beta r} - R_r)}{P^2}, \mathrm{K}_6 = -p \mathrm{K}_7 + p \mathrm{K}_8$$

$$\mathrm{K}_7 = \frac{\widehat{T}(\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta) e^{-p} (\pi_2 - 12 \Delta_2) + (R_{\alpha r} + R_{\beta r} - R_r)(\Delta_3 \pi_2 - \Delta_2 \pi_3)}{\Delta_2 \pi_1 - \Delta_1 \pi_2}$$

$$\mathrm{K}_8 = \frac{\widehat{T}(\widehat{T} R - \widehat{S_\alpha} R_\alpha - \widehat{S_\beta} R_\beta) e^{-p} (12 \Delta_1 - \pi_1) + (R_{\alpha r} + R_{\beta r} - R_r)(\Delta_1 \pi_3 - \Delta_3 \pi_1)}{\Delta_2 \pi_1 - \Delta_1 \pi_2}$$

VI. COMPATIBILITY CONDITION

The differential equations corresponding to concentration and temperature and corresponding boundary conditions give the compatibility condition as below.

$$\left\{\lambda_4 \int_0^1 W_{\alpha 1}(z)dz + \lambda_5 \int_0^1 W_{\alpha r1}(z_r)dz_r\right\} = \widehat{T} + 2\widehat{d}^{\,2} + \widehat{S_\beta} \quad (51)$$

Now to find Critical Rayleigh number substitute (49) and (50) in equation (51) then solving for the critical Rayleigh number, we get

$$R = \frac{(2\widehat{d}^2+\widehat{T}+\widehat{S_\beta}) + \mathrm{R}_\alpha\,(\widehat{S}_\alpha X_2 - X_3\lambda_1) + \mathrm{R}_\beta\,(\widehat{S}_\beta X_2 - X_3\lambda_2)}{(\widehat{T}\,X_2 - X_3\lambda_3)}$$

Where,

$$\boldsymbol{\delta_1} = \frac{\lambda_4}{24\,\widehat{T}\,\widehat{d}\beta^2}, \boldsymbol{\delta_2} = \frac{\lambda_4\,\widehat{\mu}}{6\widehat{T}}, \quad \boldsymbol{\delta_3} = \frac{\lambda_4\widehat{d}}{2\,\widehat{T}}, \boldsymbol{\delta_4} = \frac{\widehat{d}^2\lambda_4}{\widehat{T}}, \boldsymbol{\eta_1} = \boldsymbol{pe^{-p}(1-p-e^{p})},\ \boldsymbol{\eta_2 = e^{-p}p^2}$$

$$\boldsymbol{X_2 = \lambda_4 + A_1\widehat{T}\,e^{-p}(\pi_2 - 12\,\Delta_2)X_1 + A_2\widehat{T}e^{-p}(12\,\Delta_1 - \pi_1)X_1, \eta_3 = p\,(1-e^{-p})}$$

$$\boldsymbol{A_1} = \sum_{k=1}^{4}\boldsymbol{\delta_k\eta_k - \lambda_5}\left(\boldsymbol{1} + \frac{p}{2} - \left(\frac{e^{p}-1}{p}\right)\right), \boldsymbol{A_2} = \sum_{k=1}^{4}\boldsymbol{\delta_k\gamma_k - \lambda_5}\left(\boldsymbol{1} - \frac{2p}{4} - \left(\frac{-1+e^{-p}}{p}\right)\right)$$

$$\boldsymbol{\gamma_1} = \frac{p[(p+e^{-p})-1]}{e^{-p}}, \boldsymbol{\gamma_2} = \frac{e^{-p}\,p^2}{e^{-2p}}, \boldsymbol{\gamma_3} = \frac{p\,(1-e^{-p})}{e^{-p}}, \boldsymbol{\gamma_4} = \frac{1-pe^{-p}-e^{-p}}{e^{-p}}$$

$$\boldsymbol{X_3 = A_1X_1(\Delta_3\pi_2 - \Delta_2\pi_3) + A_2(\Delta_1\pi_3 - \Delta_3\pi_1)X_1 + A_3}, \qquad \boldsymbol{\eta_4 = p - 1 + e^{-p}}$$

$$\boldsymbol{A_3 = -\delta_1 + \delta_2 + \delta_3} + \frac{\delta_4}{2} + \frac{\lambda_5}{6}, \boldsymbol{\lambda_5} = \left[\widehat{\boldsymbol{d}}^2 + \frac{\widehat{d}^2}{\tau_{\beta r}} - \frac{D_\beta\widehat{d}^2}{\tau_{\alpha r}}\right], \boldsymbol{P_1} = e^{-p} - \boldsymbol{1}$$

$$\boldsymbol{\lambda_1} = \frac{\mathrm{B}^2\widehat{d}^3 S_\beta}{\widehat{k}}, \boldsymbol{\lambda_2} = \frac{\mathrm{B}^2\widehat{d}^3 S_\alpha}{\widehat{k}}, \boldsymbol{\lambda_3} = \frac{\mathrm{B}^2\widehat{d}^3\widehat{T}}{\widehat{k}}, \boldsymbol{\lambda_4} = \left(\boldsymbol{1} - \frac{D_\alpha}{\tau_\alpha} + \frac{1}{\tau_\beta}\right)\frac{1}{120}, \boldsymbol{X_1} = \frac{1}{\Delta_2\,\pi_1 - \Delta_1\,\pi_2}$$

$$\boldsymbol{\Delta_1 = 24\,\widehat{d}^2(1 - e^{-p}(p-1)) + 24\,\widehat{d}\,p\,(-e^{-p}+1) + 12\,\mu\,p^2} + \frac{4p\ (p+P_1)}{\widehat{d}\mathrm{B}^2}$$

$$\boldsymbol{\Delta_2 = 24\widehat{d^2}(P_1+p)e^{-p} + 24pe^{-p}(1-e^{-p}) + 12\widehat{\mu}p^2e^{-2p}} + \frac{4pe^{-2p}(1-p-e^{p})}{d\mathrm{B}^2}$$

$$\boldsymbol{\Delta_3 = 12\widehat{d}^2e^{-p} + 24\,\widehat{d}\,e^{-p} + 12\,\widehat{\mu}\,e^{-p}} - \frac{4e^{-p}}{\widehat{d}\,\mathrm{B}^2},$$

$$\boldsymbol{\pi_3 = 24\widehat{d}e^{-p} + 24\widehat{\mu}e^{-p}} - \frac{12e^{-p}}{\widehat{d}\mathrm{B}^2}$$

$$\boldsymbol{\pi_1 = 24\widehat{d}p(1-e^{-p}) + 24\,\widehat{\mu}\,p^2} + \frac{12p(P_1+p)}{\widehat{d}\mathrm{B}^2},\ \boldsymbol{p} = (\widehat{\mu}\,\mathrm{B}^2)^{-1/2},\ \mathbf{B} = \sqrt{\mathbf{Da}}$$

$$\boldsymbol{\pi_2 = e^{-p}\,24\widehat{d}p(1-e^{-p}) + 24\widehat{\mu}p^2e^{-2p}} + \frac{12pe^{-2p}(1-p-e^{p})}{\widehat{d}\mathrm{B}^2}$$

VII. RESULTS AND DISCUSSIONS

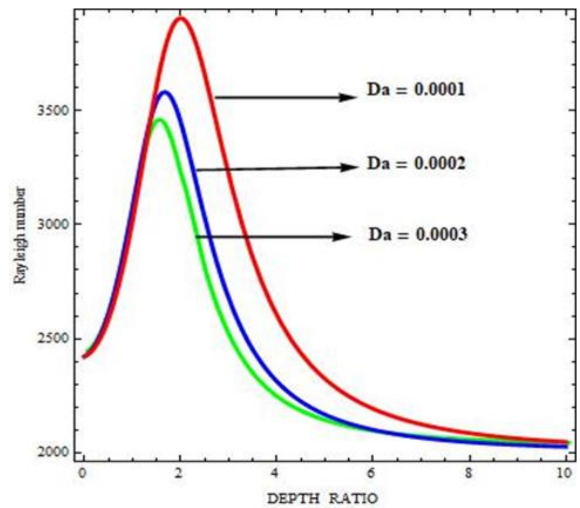


Fig. 1. effect of $\mathcal{D}_a$ on $\mathcal{R}$ayleigh Number R

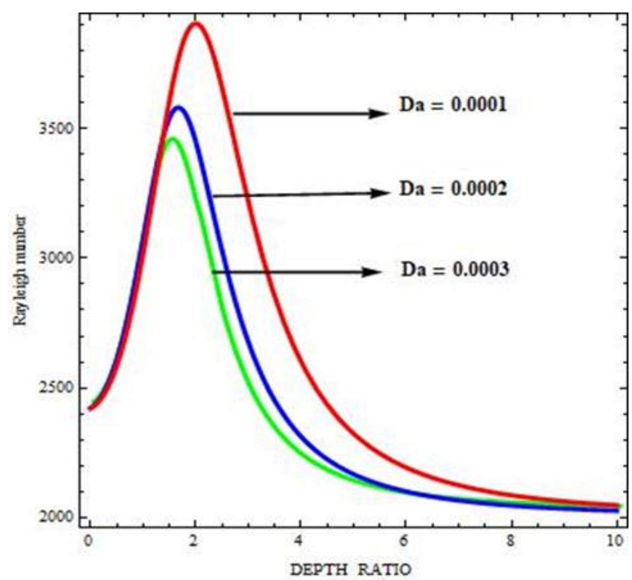


Fig. 2. effect of $\mathcal{D}_a$ on $\mathcal{R}$ayleigh Number R

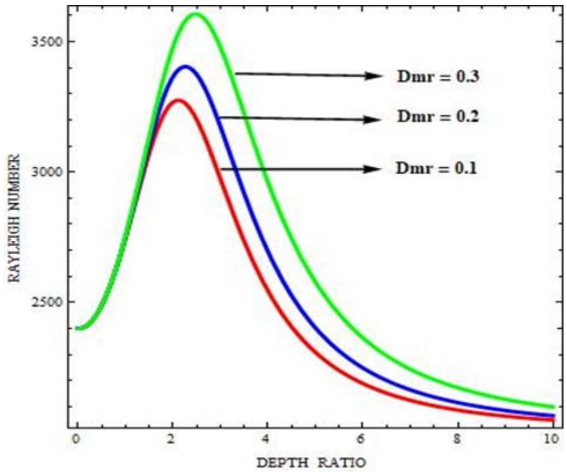


Fig. 3. effect of $\mathcal{D}_{mr}$ on $\mathcal{R}$ayleigh Number R

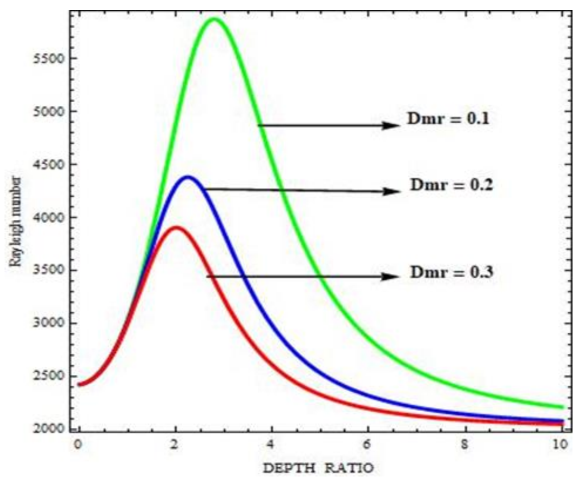


Fig. 4. effect of $\mathcal{D}_{mr}$ on $\mathcal{R}$ayleigh Number R

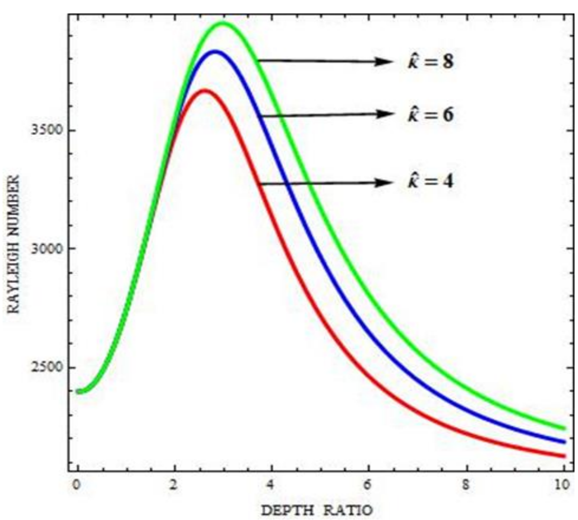


Fig. 5. Effect of $\hat{\kappa}$ on $\mathcal{R}$ayleigh Number R

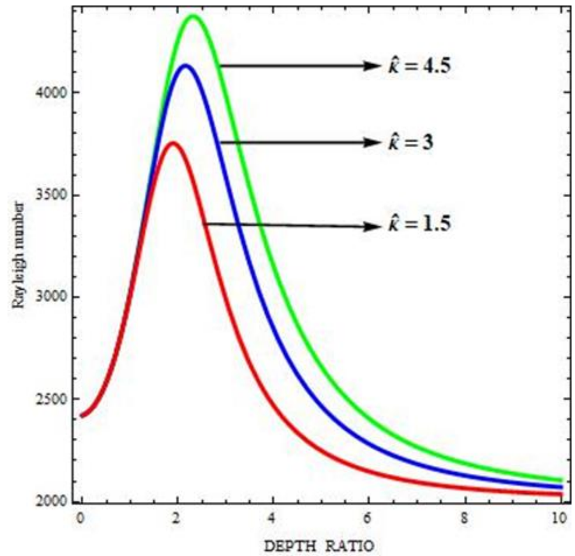


Fig. 6. Effect of $\hat{\kappa}$ on $\mathcal{R}$ayleigh Number R

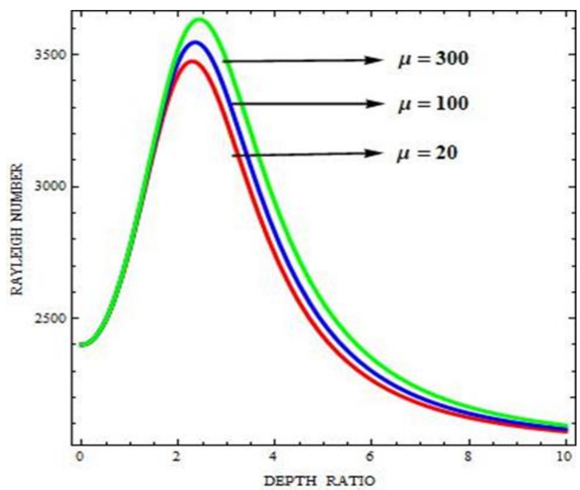


Fig. 7. Effect of μ on $\mathcal{R}$ayleigh Number R

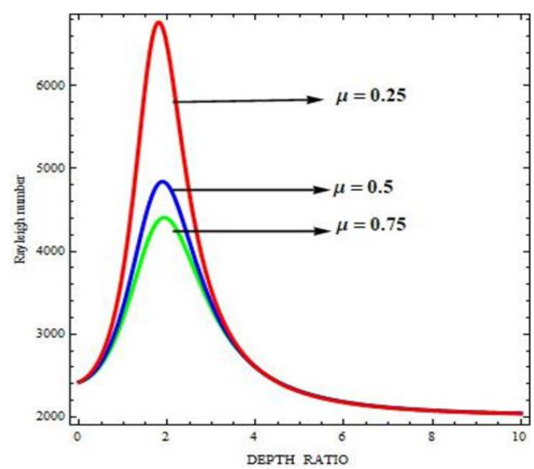


Fig. 8. Effect of μ on $\mathcal{R}$ayleigh Number R

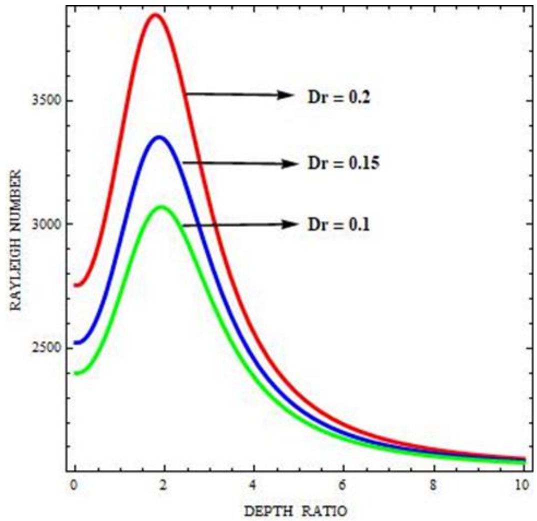


Fig. 9. effect of $\mathcal{D}_r$ on $\mathcal{R}$ayleigh Number R

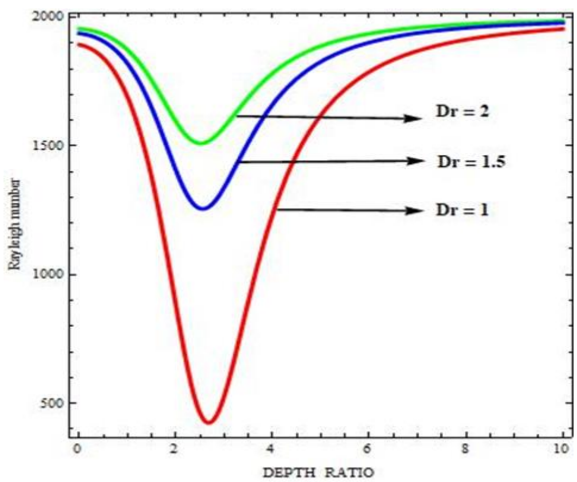


Fig. 10. effect of $\mathcal{D}_r$ on $\mathcal{R}$ayleigh Number R

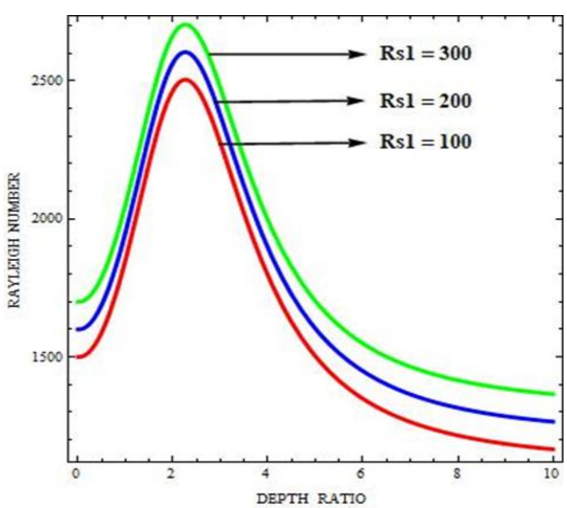


Fig. 11. Effect of R_{s1} on $\mathcal{R}$ayleigh Number R

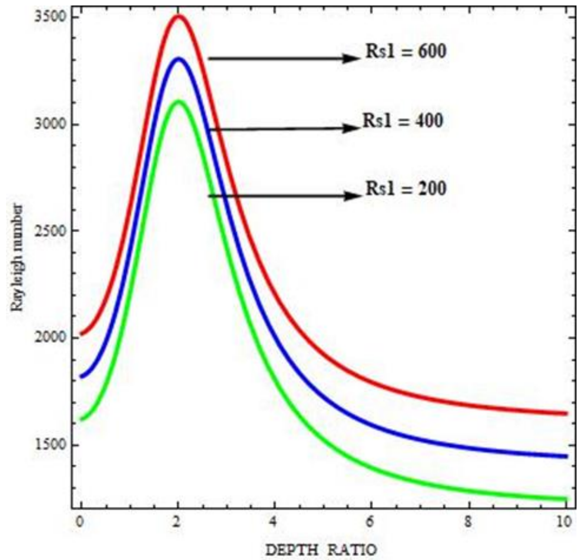


Fig. 12. Effect of R_{s1} on $\mathcal{R}$ayleigh Number R

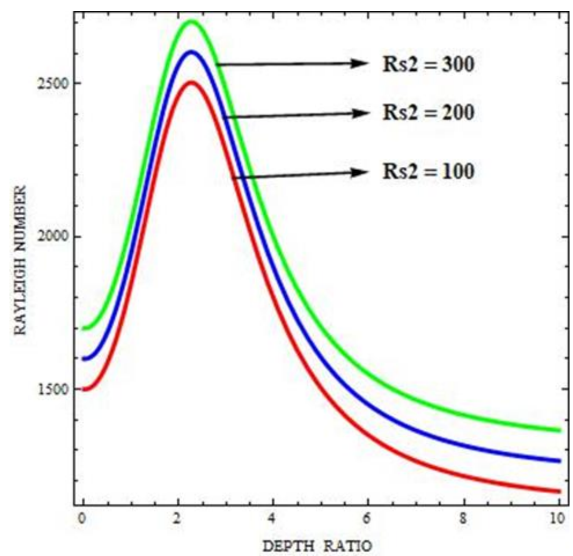


Fig. 13. Effect of R_{s2} on $\mathcal{R}$ayleigh Number R

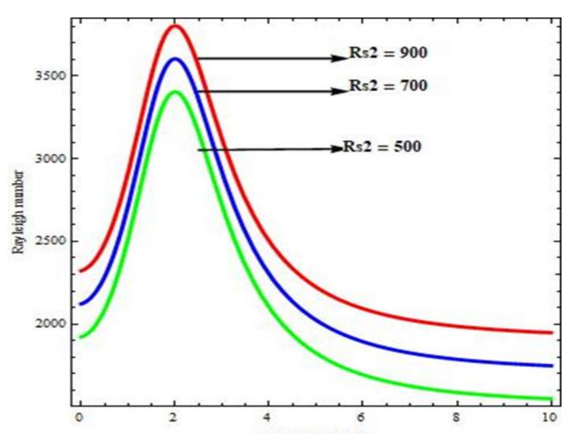


Fig. 14. Effect of R_{s2} on $\mathcal{R}$ayleigh Number R

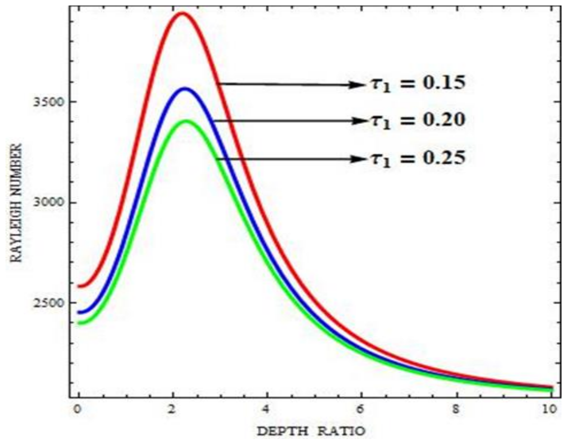


Fig. 15. Effect of τ_1 on $\mathcal{R}$ayleigh Number R

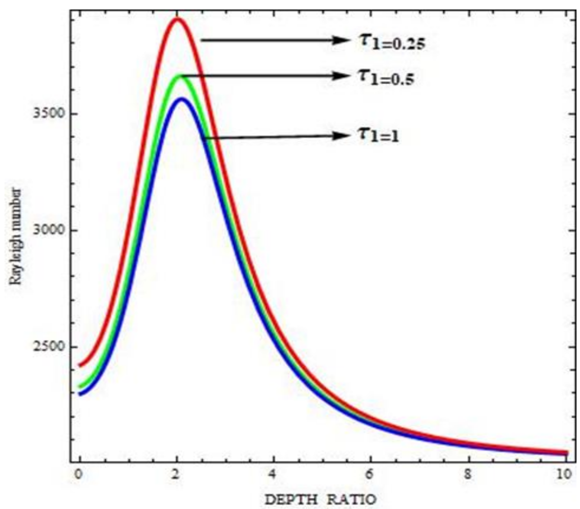


Fig. 16. Effect of τ_1 on $\mathcal{R}$ayleigh Number R

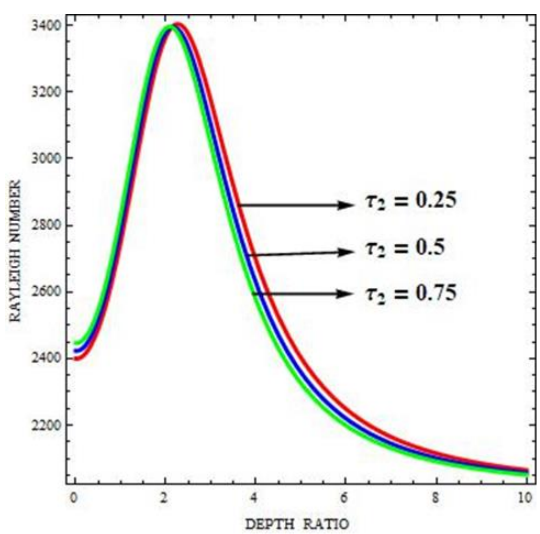


Fig. 17. Effect of τ_2 on $\mathcal{R}$ayleigh Number R

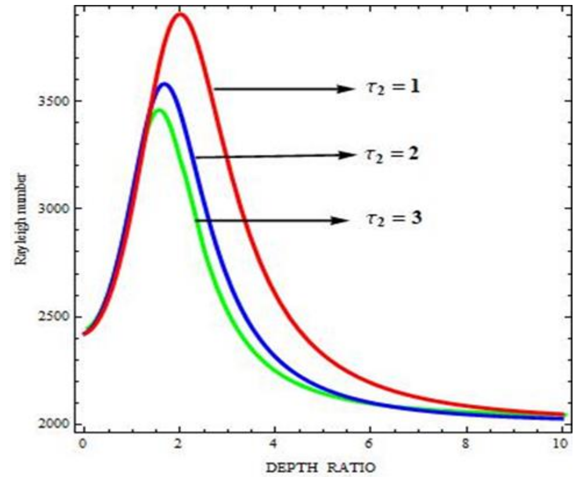


Fig. 18. Effect of τ_2 on $\mathcal{R}$ayleigh Number R

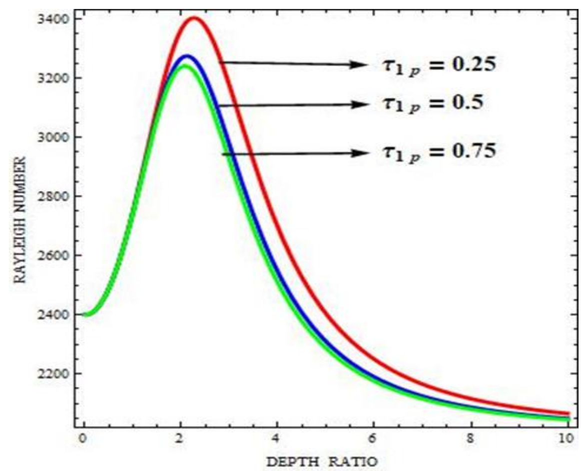


Fig. 19. Effect of τ_{1p} on $\mathcal{R}$ayleigh Number R

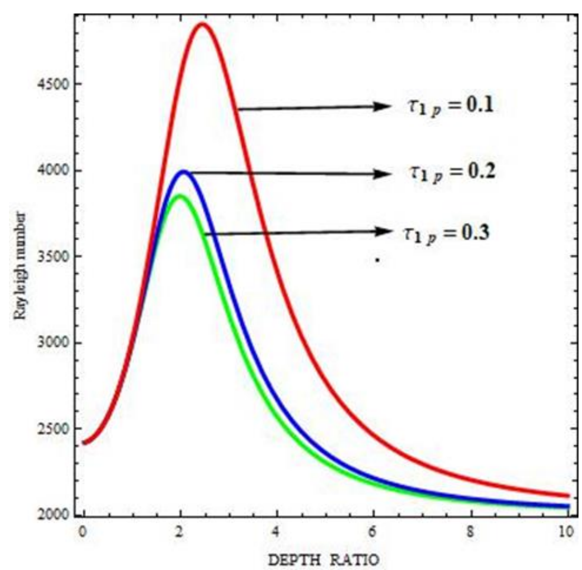


Fig. 20. Effect of τ_{1p} on $\mathcal{R}$ayleigh Number R

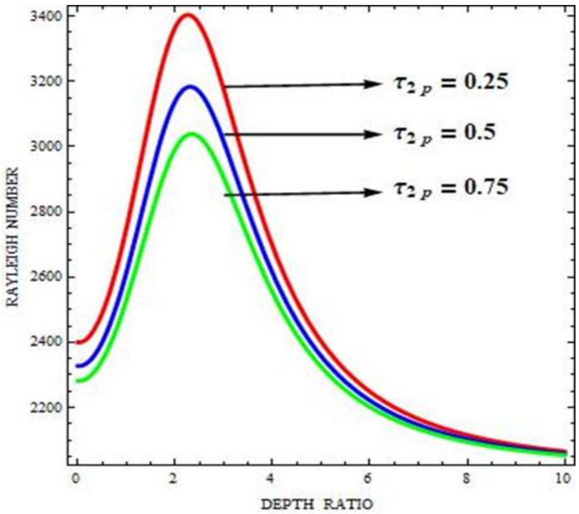


Fig. 21. Effect of τ_{2p} on $\mathcal{R}$ayleigh Number R

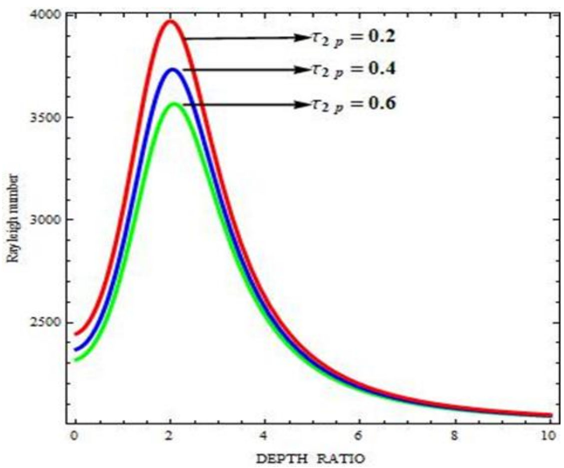


Fig. 22. Effect of τ_{2p} on $\mathcal{R}$ayleigh Number R

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