Harjeet's Synergistic Trigonometric Identities

Harjeet Singh
Sri Chaitanya Techno School

Abstract. In this paper, we present three novel trigonometric formulae derived by summing two or more trigonometric ratios. These formulae offer a new perspective on classical trigonometric relationships, providing simplified expressions for specific combinations of angles. The derivations are grounded in fundamental trigonometric identities, and their validity is demonstrated through various proofs and applications. This discovery contributes to the broader understanding of trigonometric functions and offers potential for application in advanced mathematical problems and geometrical computations. The formulae are explored in both theoretical and practical contexts, highlighting their relevance in modern mathematical analysis.

Keywords. Trigonometric Identities, Trigonometric Relations, Trigonometric Simplification, Trigonometric Equations

I. INTRODUCTION

Trigonometry, a branch of mathematics that deals with the relationships between the angles and sides of triangles, has been a fundamental tool in various fields such as physics, engineering, and astronomy. Over centuries, numerous identities and formulae have been derived, forming the backbone of trigonometric theory. These relationships, such as the sum and difference identities, double and half-angle formulas, and product-to-sum identities, provide essential methods for simplifying complex expressions and solving real-world problems. [1], [12]

Despite the vast collection of known trigonometric identities, there remains room for exploration and discovery within this rich mathematical framework. In this paper, we introduce three new trigonometric formulae derived by summing two or more trigonometric ratios. These formulae were developed because of re-examining existing relationships between trigonometric functions and seeking new ways to simplify and combine these functions.

The motivation for this work stems from the need to explore novel expressions that can offer more efficient solutions to complex problems involving angles and their functions. By expanding upon the current body of knowledge, these new formulae have the potential to enhance both theoretical and practical applications of trigonometry.

This paper will first present the derivation of the new formulae, followed by rigorous proofs and discussions on their implications. We will then explore possible applications of these formulae in solving trigonometric equations and simplifying expressions in higher-level mathematics. Through this, we aim to contribute to the ongoing development of trigonometric theory and its applications in modern mathematics.

II. LITERATURE REVIEW

Trigonometry, with its origins in ancient Greek and Indian mathematics, has long been a subject of intense study and application. The foundational works of scholars like Hipparchus, Ptolemy, Aryabhata, and Bhaskara have established many of the trigonometric identities and principles still in use today. Over time, these classical principles have evolved, with modern mathematicians developing more advanced identities and applications. [1], [2], [12]

One of the earliest and most fundamental results in trigonometry is the Pythagorean identity, which has been the basis for numerous discoveries over the centuries. Following this, sum and difference formulas, double and half-angle identities, and product-to-sum transformations were derived and widely applied in both pure and applied mathematics. These identities simplify trigonometric expressions and help solve a wide range of problems, from calculating angles in geometry to solving complex differential equations in physics. [3]

The *sum of trigonometric ratios* has been particularly significant in areas such as Fourier analysis and signal processing, where understanding the relationships between trigonometric functions is essential. However, despite the abundance of identities derived over the years, the exploration of *sums involving multiple trigonometric ratios* remains an area of ongoing research. [4], [5] In recent years, various scholars have contributed to extending classical identities. For instance, generalized formulas for the sum and product of multiple angles have emerged, offering deeper insights into the relationships between trigonometric functions. Notably, the work on sum and difference identities has led to more elegant formulations in calculus, geometry, and number theory, illustrating the power of combining multiple trigonometric functions.

However, there is limited research specifically focused on discovering new identities that sum two or more trigonometric ratios. Much of the current literature focuses on refining existing relationships rather than discovering new ones. This gap highlights the need for further exploration, particularly in deriving novel and useful formulas that can provide more streamlined approaches to solving trigonometric problems.

The current study builds upon this foundation by introducing new trigonometric formulae derived from summing multiple trigonometric ratios. These new expressions offer potential improvements over existing methods, especially in cases where traditional formulas are less efficient or more complex. By examining these newly derived identities, this research aims to contribute to the ongoing development of trigonometric theory, providing valuable tools for both theoretical and practical applications.

III. DERIVATIONS AND PROOFS

Let there be a triangle ΔXYZ where a is perpendicular, b is base and c is hypotenuse.

Formula 1. $cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$

Derivation. Let us sum two trigonometric functions $cosec\,\theta$ and $sec\,\theta$. [4], [5]

$$cosec\,\theta + sec\,\theta$$

$$\frac{c}{a} + \frac{c}{b}$$

$$\frac{cb + ca}{ab}$$

$$\frac{c(a+b)}{ab}$$

$$\frac{c}{a}\left(\frac{a+b}{b}\right)$$

$$\frac{c}{a}\left(\frac{a}{b} + 1\right)$$

We Know that,

$$cosec\theta = \frac{c}{a}$$

$$tan\,\theta = \frac{a}{b}$$

Substituting this value in the expression,

$$cosec\,\theta(tan\,\theta + 1)$$

As this has been derived by the sum of $cosec\,\theta$ and $sec\,\theta$. These expressions should be equal,

$$cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$$

Proof 1. To Prove:
[5], [6]

$$cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$$

Simplifying Right Hand Side,

$$cosec\,\theta(tan\,\theta + 1)$$

$$\frac{1}{sin\theta}\left(\frac{\sin\theta}{\cos\theta} + 1\right)$$

$$\frac{1}{sin\,\theta} + \frac{1}{cos\,\theta}$$

$$cosec\,\theta + sec\,\theta$$

LHS=RHS

Hence Proved

Proof 2. To Prove:
[5], [6]

$$cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$$

Simplifying Right Hand Side,

$$cosec\,\theta(tan\,\theta + 1)$$

$$cosec\theta\left(\frac{\sin\theta}{\cos\theta} + 1\right)$$

$$\frac{1}{sin\theta}\left(\frac{\sin\theta + cos\theta}{\cos\theta}\right)$$

$$\frac{\sin\theta + cos\theta}{\cos\theta\sin\theta}$$

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta}$$

$$cosec\,\theta + sec\,\theta$$

LHS=RHS

Hence Proved

Proof 3. To Prove:
[5], [6], [7]

$$cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$$

Let us assume,

$$f(\theta) = cosec\,\theta + sec\,\theta$$

$$g(\theta) = cosec\,\theta(tan\,\theta + 1)$$

Differentiating both functions,

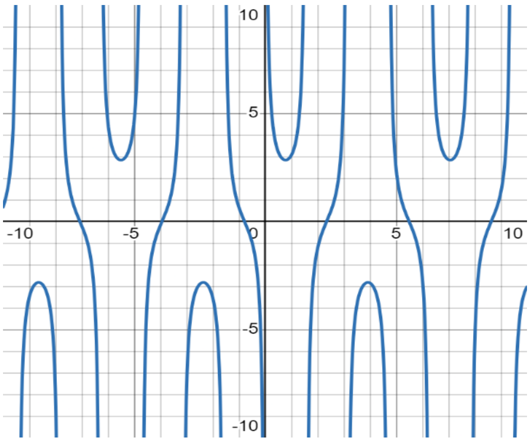
$$f'(\theta) = -cosec\theta \cot\theta - \sec\theta \tan\theta$$

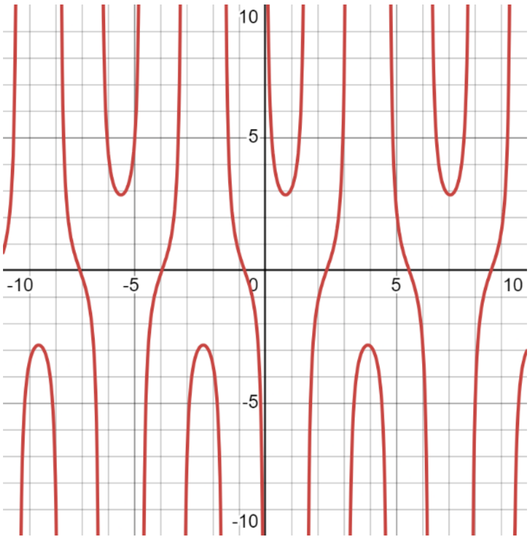
$$g'(\theta) = cosec\theta sec^2\theta - cosec\theta \tan\theta$$

For this identity to hold,

$$f'(\theta) = g'(\theta)$$

To illustrate the equality let us plot the graph of both functions,





These graphs illustrate the functions $f(\theta) = cosec\,\theta + sec\,\theta$ which is represented by blue line and $g(\theta) = cosec\,\theta(tan\,\theta + 1)$ is represented by red line.

The identical nature of the two functions visually demonstrates their equality, supporting the proof that $f(\theta) = g(\theta)$.

Formula 2. $sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\,\theta) + sec\,\theta$

Derivation. Let us sum three trigonometric ratios $\sin\theta$, $\sec\theta$ and $\tan\theta$. [4], [5]

$$\sin\theta + \sec\theta + \tan\theta$$

$$\frac{a}{c} + \frac{c}{b} + \frac{a}{b}$$

$$\frac{ab + c^2 + ac}{cb}$$

$$\frac{a(b + c) + c^2}{cb}$$

$$\frac{a[b + c]}{cb} + \frac{c^2}{cb}$$

$$\frac{a}{c}\left(1 + \frac{c}{b}\right) + \frac{c}{b}$$

We Know That,

$$\sin\theta = \frac{a}{c}$$

$$\sec\theta = \frac{c}{b}$$

Substituting this value in the equation,

$$sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\,\theta) + sec\,\theta$$

Proof 1. To Prove:
[5], [6]

$$sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\theta) + sec\theta$$

Simplifying LHS,

$$sin\,\theta + sec\,\theta + tan\,\theta$$

$$\sin\theta + \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta + \frac{1 + \sin\theta}{\cos\theta}$$

Simplifying RHS,

$$sin\,\theta(1 + sec\theta) + sec\theta$$

$$sin\,\theta\left(1+\frac{1}{cos\,\theta}\right)+\frac{1}{cos\,\theta}$$

$$\sin\theta+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}$$

$$\sin\theta+\frac{1+\sin\theta}{\cos\theta}$$

LHS=RHS

Hence Proved

Proof 2. To Prove:
[5], [6]

$$sin\,\theta+sec\,\theta+tan\,\theta=sin\,\theta(1+sec\theta)+sec\theta$$

Let us assume,

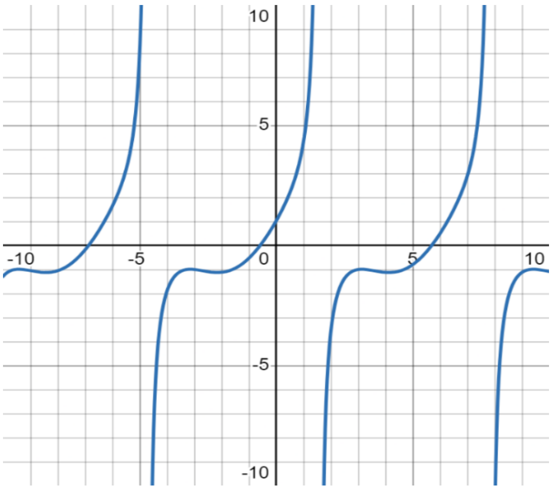
$$f(\theta)=sin\,\theta+sec\,\theta+tan\,\theta$$

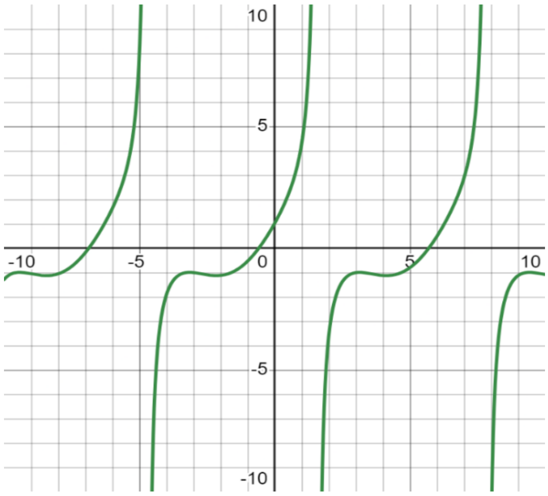
$$g(\theta)=sin\,\theta(1+sec\theta)+sec\theta$$

For this identity to hold,

$$f(\theta)=g[\theta]$$

Examining this equality through graph,





Here the blue line represents $f[\theta]$ and the green line represents $g[\theta]$.

The identical nature of the two functions visually demonstrates their equality, supporting the proof that $f(\theta)=g(\theta)$.

Proof 3. To Prove:
[5], [6], [8]

$$sin\,\theta+sec\,\theta+tan\,\theta=sin\,\theta(1+sec\theta)+sec\theta$$

Let us assume,

$$f(\theta)=sin\,\theta+sec\,\theta+tan\,\theta$$

$$g(\theta)=sin\,\theta(1+sec\theta)+sec\theta$$

Integrating both sides,

$$\int f(\theta)\;d\theta=-\cos\theta+\ln\left|\frac{\sec\theta+\tan\theta}{\cos\theta}\right|+C_1$$

$$\int g(\theta)\,d\theta=-\cos\theta+\ln\left|\frac{\sec\theta+\tan\theta}{\cos\theta}\right|+C_2$$

Since both integrals are equal except for the constants C_1 and C_2, we can conclude that:

$$\int f(\theta)\,d\theta=\int g(\theta)\,d\theta$$

By this expression we can conclude that,

$$f(\theta)=g[\theta]$$

LHS=RHS

Hence Proved.

Formula 3. $cot\,\theta+sin\,\theta=cos\,\theta\,(tan\,\theta+cosec\,\theta)$

Derivation. Let us sum two trigonometric ratios $\cot\theta$ and $\sin\theta$. [4], [5]

$$\cot\theta+\sin\theta$$

$$\frac{b}{a}+\frac{a}{c}$$

$$\frac{bc+a^2}{ac}$$

To simplify this, we need to examine $\tan\theta+\csc\theta$,

$$\tan\theta+\csc\theta=\frac{a^2+bc}{ab}$$

By this equation,

$$\frac{bc + a^2}{ac} = \frac{b}{c}(\tan\theta + \csc\theta)$$

We Know that,

$$\cos\theta = \frac{b}{c}$$

Substituting this value in the equation,

$$\frac{bc + a^2}{ac} = \cos\theta\,(\tan\theta + \csc\theta)$$

$$\cot\theta + \tan\theta = \cos\theta\,(\tan\theta + \csc\theta)$$

Proof 1. To Prove:
[5], [6]

$$\cot\theta + \sin\theta = \cos\theta\,(\tan\theta + \csc\theta)$$

Simplifying RHS,

$$\cos\theta\,(\tan\theta + \csc\theta)$$

$$cos\,\theta \cdot \frac{sin\,\theta}{cos\,\theta} + cos\,\theta \cdot \frac{1}{sin\,\theta}$$

$$\sin\theta + \cot\theta$$

LHS=RHS

Hence Proved [5],

Proof 2. To Prove:
[5], [6], [9]

$$\cot\theta + \sin\theta = \cos\theta\,(\tan\theta + \csc\theta)$$

We Know that,

$$\cot\theta + \sin\theta = \frac{\cos\theta + sin^2\theta}{\sin\theta}$$

$$\cos\theta\,(\tan\theta + \csc\theta) = \frac{\cos\theta[1 + \sin\theta]}{\sin\theta}$$

Establishing Equality using Limits,

$$\lim_{\theta\to 0}\frac{\cos\theta + sin^2\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1}{\theta} + \theta$$

$$\lim_{\theta\to 0}\frac{\cos\theta[1 + \sin\theta]}{\sin\theta}$$

Equating the limits,

$$\lim_{\theta\to 0}\frac{1}{\theta} + \theta \to \infty$$

$$\lim_{\theta\to 0}\frac{\cos\theta[1 + \sin\theta]}{\sin\theta} \to \infty$$

Both sides yield the same limit behaviour.

Thus, we conclude that both sides of the equation approach the same value as θ approaches 0:

$$\cot\theta + \sin\theta = \cos\theta\,(\tan\theta + \csc\theta)$$

Proof 3. To Prove:
[5], [6], [8]

$$\cot\theta + \sin\theta = \cos\theta\,(\tan\theta + \csc\theta)$$

Integrating Both sides,

$$\int [\cot\theta + \sin\theta]\ d\theta = \ln|\sin\theta| - \cos\theta + C$$

$$\int [\cos\theta\,(\tan\theta + \csc\theta)]\ d\theta = -\cos\theta + \ln|\sin\theta| + C$$

Since both integrals yield the same result,

Thus, the original identity holds true.

IV. DISCUSSION AND APPLICATIONS

The three derived trigonometric identities — $cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$, $sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\theta) + sec\theta$, and $\cot\theta + \tan\theta = \cos\theta\,(\tan\theta + \csc\theta)$— present significant advancements in understanding the relationships among trigonometric functions. Each formula offers a new perspective on how these functions can be expressed in more compact and interconnected forms, revealing the elegant structure of trigonometric relationships.

Significance of the Formulae These identities underscore the power of simplification in mathematical expressions, particularly in trigonometry. By consolidating multiple trigonometric functions into more unified forms, these formulae streamline calculations and enhance our understanding of trigonometric relationships. This is especially valuable when solving complex equations or when conducting analyses in fields such as geometry, physics, and engineering. [10]

Comparison with Existing Identities. These identities can be compared to traditional trigonometric identities such as the Pythagorean identity and sum/difference formulas. However, the introduction of terms like secant, tangent, and cosecant in novel configurations highlights new ways to approach trigonometric manipulation. The inclusion of functions like secant and cosecant makes these formulae versatile in solving real-world problems that involve trigonometric functions.

Applications. The derived formulae find practical application across multiple domains: [11]

- *Solving Trigonometric Equations:* These identities simplify the process of solving complex equations involving sine, cosine, tangent, and other trigonometric functions. They offer more straightforward pathways to isolating variables and finding solutions.
- *Geometric Problems:* In geometry, these identities can assist in solving problems involving right triangles, circles, and angles. They simplify the relationships between angles and side lengths, which are critical in geometric computations.
- *Physics and Engineering:* In physics, where wave motion, oscillations, and rotational dynamics are frequently analysed, these identities provide easier ways to compute forces and trajectories. In engineering, they assist in designing systems that rely on trigonometric relationships.
- *Calculus Applications:* The formulae facilitate both differentiation and integration of trigonometric functions. Their simplified forms make it easier to work with integrals and derivatives involving secant, cosecant, and tangent functions, reducing the complexity of calculus problems.
- *Computer Graphics and Signal Processing*: These formulae are useful in fields such as computer graphics and signal processing, where trigonometric functions model periodic motion, rotations, and transformations. The identities streamline calculations, improving computational efficiency.

Limitations and Future Research. While these formulae are robust and versatile, they are only valid for angles where the involved trigonometric functions are defined. Future research could focus on extending these identities to other trigonometric functions or their hyperbolic counterparts. Additionally, exploring their applications in higher dimensions or in advanced mathematical theories such as Fourier analysis or complex analysis could open new avenues of discovery.

Conclusion of Discussion and Application. The three newly derived trigonometric identities —$cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$, $sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\theta) + sec\theta$ and $\cot\theta + \tan\theta = \cos\theta\,(\tan\theta + \csc\theta)$— significantly contribute to the existing body of knowledge on trigonometric relationships. They simplify complex expressions and offer new insights into the interplay between different trigonometric functions. Their applications span various fields, from geometry and calculus to physics, engineering, and computer science, showcasing the continued relevance and utility of trigonometry in solving real-world problems.

V. CONCLUSION

This research has introduced and validated three significant trigonometric identities:

1. *First Formula:* $cosec\,\theta + sec\,\theta = cosec\,\theta(tan\,\theta + 1)$
2. *Second Formula:* $sin\,\theta + sec\,\theta + tan\,\theta = sin\,\theta(1 + sec\theta) + sec\theta$
3. *Third Formula:* $\cot\theta + \tan\theta = \cos\theta\,(\tan\theta + \csc\theta)$

Each of these identities highlights the intricate relationships between the sine, secant, and tangent functions, offering a fresh perspective on their interdependencies.

Significance of the Findings

The derived formulas contribute valuable insights into the realm of trigonometry by simplifying complex expressions and showcasing the interconnectedness of various trigonometric functions. These identities not only facilitate easier manipulation and computation but also enhance our understanding of fundamental trigonometric relationships.

Practical Applications

The applications of these identities extend across multiple fields, including mathematics, physics, engineering, and computer graphics. By simplifying the analysis of trigonometric equations, these formulas enhance problem-solving efficiency in geometric contexts, wave mechanics, and calculus. Their versatility makes them useful tools for both theoretical exploration and practical implementation. [11]

Future Directions

While this research contributes to the understanding of trigonometric identities, it also opens the door for further exploration. Future studies could investigate additional identities, potential generalizations, or applications in advanced mathematical fields such as complex analysis and signal processing. The ongoing search for new relationships within trigonometry remains an exciting area for mathematical inquiry.

Final Remarks

In conclusion, the derived trigonometric identities embody the elegance and complexity of trigonometric functions. They not only serve as useful tools for mathematical problem-solving but also deepen our appreciation for the interconnectedness of mathematical concepts. As the exploration of trigonometric relationships continues, these identities will undoubtedly inspire further discoveries and applications in various disciplines.

VI. ACKNOWLEDGMENTS

I would like to express my sincere gratitude to everyone who has supported me throughout this research.

First and foremost, I extend my appreciation to Mr. Harbhajan Singh and Dr. Manjit Kaur whose invaluable insights and encouragement played a significant role in shaping the direction of this work. Their constant guidance was instrumental in completing this project.

I would also like to thank Mr. Harinder Singh for their thoughtful discussions and feedback, which greatly contributed to refining the ideas presented in this paper.

Additionally, I acknowledge the use of Desmos as the primary graphing tool in this research. Its intuitive interface and robust functionality allowed for precise visualizations and enhanced the understanding of complex mathematical relationships.

Finally, I am grateful to Ms. Satinder Kaur for their unwavering support and belief in my abilities, which motivated me to push through the challenges.

REFERENCES

[1]. Adamek, Tara, Kaitlyn Penkalski, and Gina Valentine. "The history of trigonometry." *History of Mathematics* 11 [2005].

[2]. Duke, Dennis W. "The Very Early History of Trigonometry." *International Journal of Scientific History* 17 [2011]: 34-42.

[3]. Ellermeyer, S. "Trigonometric Identities." *Department of Mathematics. Recuperado de: http://facultyweb. kennesaw. edu/sellerme/docs/basictrigidentities. pdf* [2014].

[4]. Flanders, Harley, and Justin J. Price. *Algebra and trigonometry*. Elsevier, 2014.

[5]. McKeague, Charles P. *Intermediate algebra with trigonometry*. Academic press, 2014.

[6]. Liu, Zhou, et al. "Learning to prove trigonometric identities." *arXiv preprint arXiv:2207.06679* [2022].

[7]. Kishan, Hari. *Differential calculus*. Atlantic Publishers & Dist, 2007.

[8]. Integral Formulas - List of All Integral Formulas." *Byju's*, https://byjus.com/integral-formulas/. Accessed 2 Oct. 2024.

[9]. Cornu, Bernard. "Limits." *Advanced mathematical thinking*. Dordrecht: Springer Netherlands, 1991. 153-166.

[10]. Hedrick, E. R. "The significance of mathematics." *The American Mathematical Monthly* 24.9 [1917]: 401-406.

[11]. Vijayaram, T. R., and N. Ramya. "Applications of Trigonometry In Engineering And Technology." *International Journal of Mechanics and Finite Elements* 10.1 [2024]: 11-15p.

[12]. Ibragimov, N. Sh, and J. A. Kattaxo'jaeva. "History of trigonometry." *Miasto Przyszłości* 24 [2022]: 37-43.