© October 2024 | IJIRT | Volume 11 Issue 5 | ISSN: 2349-6002

Fuzzy Soft Vector Space

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Abstract: This paper defines a definition of fuzzy soft vector spaces (FSVS) and the concept of fuzzy soft vector spaces is applied to the elementary definition of vector spaces. With the new concept fundamental notions are those of basis, dimension, span and linear dependence is developed in this paper

Index Terms: Fuzzy set, Fuzzy soft set, Fuzzy vector spaces, Fuzzy soft vector spaces.

I. INTRODUCTION

The concept of vector spaces, introduced by I.N. Herstein [13], has played a significant role in various mathematical models, particularly in geometry and physics. These spaces, which are based on the structure of an abelian group, differ from other algebraic structures by incorporating elements outside their own set, giving rise to key notions such as linear dependence, basis, and dimension.

Soft set theory, introduced by Molodtsov in 1999 [11], provides a robust framework for addressing uncertainty, surpassing traditional mathematical tools like probability theory and fuzzy set theory. This theory has found applications in decision-making, smoothness of functions, and more. However, in real-world problems, the fuzzy nature of parameters complicates the use of standard soft set theory.

In response to this, Kharal and Ahmad [6] introduced the concept of fuzzy soft sets, a significant step toward addressing complex situations where uncertainty prevails.

This paper builds on these ideas by focusing on fuzzy soft vector spaces, a natural extension of both vector spaces and fuzzy soft sets. The paper focuses on fuzzy soft vector spaces and is divided into two main sections. Section 3 introduces fundamental definitions, starting with the concept of a fuzzy soft vector space, which combines elements of fuzzy set theory and soft set theory. It also defines fuzzy soft basis, used to describe the structure of these spaces, and fuzzy soft linear span, representing all possible fuzzy soft linear combinations. The section discusses fuzzy soft finite dimensional spaces and provides examples to clarify these concepts. Section 4 explores various properties, theorems, and lemmas related to these concepts.

II. PRELIMINARIES

Definition 2.1: (Fuzzy set) [15]
Let U be a universal set A fuzzy set (class) X over U is a set characterized by a function $f_X: U \to [0, 1]$.f_X is called the membership, characteristic or indicator function of the fuzzyset X and the value $f_X(u)$ is called the grade of membership of $u \in U$ in X.

Definition 2.2: (Soft set) [11]
Let U be a universal set (space of points or objects), E be a set of parameters and $A \subseteq E$. The power set of U is defined by $P(U) = 2^U$. A pair (F, A) is called a soft set over U and is defined as a set of ordered pairs $F_A = \{(e, F_A(e)): e \in A,\ F_A(e) \in P(U)\}$, where F is a mapping given by F : A →P(U).A is called the support of F_A and we have $F_A(e) \neq \emptyset$for all $e \in A$ and $F_A(e) = \emptyset$ for all $e \notin A$.

Definition 2.3: (Fuzzy soft set) [13]
Let U be an initial universal set, E be a set of parameters and $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F: A \to F^u$, whereF^udenotes the collection of all fuzzy subsets of U (the power set of fuzzy sets on U) and the fuzzy subset of U is defined as a map f from U to [0,1]. The family of all fuzzy soft sets (F, A)over a universal set U, in which all the parameter sets A are the same, is denoted by

$FSS(U)_A = FSS(\widetilde{U})$.

Definition 2.4: (Vector space) [7]
Let V be an non-empty set and F is field. Then V is called vector space. If for every $\alpha, \beta \in F \& v, w \in V$ and $\alpha v \in V$ then V is satisfying the following conditions,

(1) V is an abelian group under addition
(2) $\alpha(v + w) = \alpha v + \alpha w$
(3) $(\alpha + \beta)v = \alpha v + \beta v$

(4) $\alpha(\beta v) = (\alpha\beta)v$
(5) $1.v = v$

(i.e.,) 1 is identity under multiplication

III. FUZZY SOFT VECTOR SPACE

Definition 3.1: (Fuzzy Soft Vector Space)
Let V_f be a fuzzy vector space over a field F and the parameter set E be the set of all real numbers $\mathbb{R}$ and A⊆E. The mapping is defined by F: A→ $P(\widetilde{U})$.
The fuzzy soft set (F, A)∈ FSV($\widetilde{U}$) is called a fuzzy soft vector over U, denoted by ($v_{fF(e)}$,A) [briefly denoted by $\tilde{v}_{fF(e)}$], if there is exactly one e∈ A such that, $f_{F(e)}(v) = \alpha$ for some $v \in U$ and $f_{F(e')}(v) = 0$ for all $e' \in A - \{e\}$.
[$\alpha \in (0,1]$ is the value of the membership degree].
The set of all fuzzy soft vectors over U is denoted by $FSV(U)_A$ = FSV($\widetilde{U}$). The set $\tilde{V}$ = FSV($\widetilde{U}$) is said to be a fuzzy soft vector space of U over F.
The set FSV($\widetilde{U}$) is a fuzzy soft vector space according to the following two operations:

i. $\tilde{v}^1_{f_1F(e_1)}+\tilde{v}^2_{f_2F(e_2)}=\left(\widetilde{v^1+v^2}\right)_{(f_1F(e_1)+f_2F(e_2))}$ for all $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)} \in \tilde{V}$
ii. $\tilde{r}.\tilde{v}_{fF(e)} = (\widetilde{rv})_{fF(re)}$, for all $\tilde{v}_{fF(e)} \in \tilde{V}$ and $\forall \tilde{r} \in \mathbb{R}(A)$.

Example 3.2:
Consider the Euclidean n-dimensional space $\mathbb{R}^n$ over $\mathbb{R}$. Let $A = \{1,2,\dots,n\}$ be the set of parameters. Let $\tilde{V}$: A→ $P(\mathbb{R}^n)$ be defined as follows:

$$\tilde{V}(i) = \left\{\tilde{v}^n_{f_nF(e_n)} \in \mathbb{R}^n(A)\right\}$$
$$\left\{i^{th}\ coordinate\ of\ \tilde{v}^n_{f_nF(e_n)}\ is\ \tilde{\theta}\right\}$$

where, $i = 1,2,\dots,n$. Then $\tilde{V}$ is a fuzzy soft vector space or a fuzzysoft linear space of $\mathbb{R}^n$ over $\mathbb{R}$.
In addition, Let $\tilde{v}^n_{f_nF(e_n)}$ be a fuzzy soft element of $\tilde{V}$ as follows:

$$\tilde{v}^n_{f_nF(e_n)} = \left(\tilde{j},\tilde{j},\dots,\tilde{j},\tilde{\theta}_{(i^{th})},\tilde{j},\dots,\tilde{j}\right) \in \mathbb{R}^n(A),$$
$i = 1,2,\dots,n$.

Then $\tilde{v}^n_{f_nF(e_n)}$ is a fuzzy soft vector of $\tilde{V}$. Where, fuzzy soft zero vector $\tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0})$ and the fuzzy soft unit vector$\tilde{j} = (\tilde{1},\tilde{1},\tilde{1},\tilde{1})$.

Definition 3.3: (Fuzzy soft subspace)
If $(v_{fF(e)}, A)$ is a fuzzy soft vector space over F and if $\tilde{w}_{fF(e)}\subset\tilde{v}_{fF(e)}$ then $\tilde{w}_{fF(e)}$ is a subspace of FSV($\widetilde{U}$).
If under the operation of $\tilde{v}_{fF(e)},\tilde{w}_{fF(e)}$itself forms a fuzzy soft vector space over F.

Definition 3.4: (Fuzzy soft linear combination)
If $(v_{fF(e)}, A)$ is a fuzzy soft vector space over F. If $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \dots, \tilde{v}^n_{f_nF(e_n)} \in \tilde{V}$ and $\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n$ ∈F then an element as of the forms,
$\tilde{\alpha}_1(\tilde{v}^1_{f_1F(e_1)})+\tilde{\alpha}_2(\tilde{v}^2_{f_2F(e_2)}) + \cdots + \tilde{\alpha}_n(\tilde{v}^n_{f_nF(e_n)}) \in \tilde{V}$is called fuzzy soft linear combination.

Definition 3.5: (Fuzzy soft linear span)
Let $(s_{fF(e)}, A)$ be a non-empty fuzzy soft subset of the FSV($\widetilde{U}$). Then the set of all linear combination of $\tilde{s}_{fF(e)}$is called linear span denoted by L($\tilde{s}_{fF(e)}$) or $<\left(\tilde{s}_{fF(e)}\right)>$.

Definition 3.6: (Fuzzy soft homomorphism)
If $(u_{fF(e)}, A)$ and $(v_{fF(e)}, A)$ are two fuzzy soft vector spaces over F, then the mapping $\tilde{T}: \tilde{u}_{fF(e)} \to \tilde{v}_{fF(e)}$ is said to be a fuzzy soft homomorphism. If,

i. $\tilde{T}\left[(\tilde{u}^1_{f_1F(e_1)}) + (\tilde{u}^2_{f_2F(e_2)})\right] = \tilde{T}(\tilde{u}^1_{f_1F(e_1)}) + \tilde{T}(\tilde{u}^2_{f_2F(e_2)})$
ii. $\tilde{T}\left[(\widetilde{\alpha u})^1_{f_1F(\alpha e_1)}\right] = \tilde{\alpha}\left[\tilde{T}(\tilde{u}^1_{f_1F(e_1)})\right]$

$\forall$ $(\tilde{u}^1_{f_1F(e_1)})$ &$(\tilde{u}^2_{f_2F(e_2)}) \in FSV(\widetilde{U})$ and $\tilde{\alpha}$ ∈F

If $\tilde{T}$ is one-to-one and onto, we call it is a fuzzy soft isomorphism.

Definition 3.7: (Fuzzy soft kernel)
The fuzzy soft kernel of fuzzy soft homomorphism $\tilde{T}: \tilde{u}_{fF(e)} \to \tilde{v}_{fF(e)}$is defined as
$ker\tilde{T} = \left\{\tilde{u}_{fF(e)} \in FSV(\widetilde{U}) / \tilde{T}(\tilde{u}_{fF(e)}) = 0\right\}$.

Definition 3.8: (Fuzzy soft finite-dimensional)
A fuzzy soft vector space$\left(v_{fF(e)}, A\right)$is said to be fuzzy soft finite-dimensional. If there is a fuzzy finite subset $\tilde{s}_{fF(e)}$ in $FSV(\widetilde{U})$ such that
$(\tilde{v}_{fF(e)} = L(\tilde{s}_{fF(e)})$.

Definition 3.9: (Fuzzy soft basis of a vector space)
Let $(s_{fF(e)}, A)$ be a fuzzy soft subset of a FSV($\widetilde{U}$), then $\tilde{s}_{fF(e)}$ is called fuzzy soft basis of $\tilde{v}_{fF(e)}$. If

i. $\tilde{s}_{fF(e)}$is fuzzy soft linearly independent.
ii. $\tilde{v}_{fF(e)} = L(\tilde{s}_{fF(e)})$

[$\left(\tilde{s}_{fF(e)}\right)$ is a fuzzy soft spanning set of FSV($\widetilde{U}$)].

Definition 3.10: (Fuzzy soft linearly dependent)
If $(v_{fF(e)}, A)$ is a fuzzy soft vector space and if $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \dots, \tilde{v}^n_{f_nF(e_n)} \in \tilde{V}$

are fuzzy soft linearly dependent over F, if there exist elements all $\tilde{\alpha}_i \neq 0 \in F$ such that,
$\tilde{\alpha}_1(\tilde{v}^1_{f_1F(e_1)}) + \tilde{\alpha}_2(\tilde{v}^2_{f_2F(e_2)}) + \cdots + \tilde{\alpha}_n(\tilde{v}^n_{f_nF(e_n)}) = 0.$

Definition 3.11: (Fuzzy soft linearly independent)
If $(v_{fF(e)}, A)$ is a fuzzy soft vector space and if, $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \dots, \tilde{v}^n_{f_nF(e_n)} \in \tilde{V}$ are fuzzy soft linearly independent over F, if there exist elements each $\tilde{\alpha}_i = 0 \in F$ such that,
$\tilde{\alpha}_1(\tilde{v}^1_{f_1F(e_1)}) + \tilde{\alpha}_2(\tilde{v}^2_{f_2F(e_2)}) + \cdots + \tilde{\alpha}_n(\tilde{v}^n_{f_nF(e_n)}) = 0.$

Definition 3.12: (Fuzzy soft quotient space)
Let $(v_{fF(e)}, A)$ be a fuzzy soft vector space over a field F and $(w_{fF(e)}, A)$ be a fuzzy soft subspace of FSV($\tilde{U}$). Then $\tilde{v}_{fF(e)}/\tilde{w}_{fF(e)}$ is a fuzzy soft vector space over F, then $\tilde{v}_{fF(e)}/\tilde{w}_{fF(e)}$ is defined by $\tilde{v}_{fF(e)} + \tilde{w}_{fF(e)} \in \tilde{v}/\tilde{w}$.

i. $[\tilde{v}^1_{f_1F(e_1)} + \tilde{w}_{fF(e)}] + [\tilde{v}^2_{f_2F(e_2)} + \tilde{w}_{fF(e)}] = (\widetilde{v^1 + v^2})_{f_1F(e_1)+f_2F(e_2)} + \tilde{w}_{fF(e)}$

ii. $\tilde{\alpha}[\tilde{v}^1_{f_1F(e_1)} + \tilde{w}_{fF(e)}] = (\widetilde{\alpha v^1})_{fF(\alpha e)} + \tilde{w}_{fF(e)}$

where $(\widetilde{v^1}_{f_1F(e_1)} + \tilde{w}_{fF(e)}), (\widetilde{v^2}_{f_1F(e_2)} + \tilde{w}_{fF(e)}) \in \tilde{v}/\tilde{w}$, $\tilde{\alpha} \in F$.
And $\tilde{v}_{fF(e)}/\tilde{w}_{fF(e)}$ is called the fuzzy soft quotient space of $\tilde{V}$ by $\tilde{W}$.

Definition 3.13: (Fuzzy soft internal direct sum)
Let $(v_{fF(e)}, A)$ be a fuzzy soft vector space over F and let $(\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$ be fuzzy soft subspaces of $\tilde{V}_{fF(e)}$.
Then if $\tilde{v}_{fF(e)}$ has one and only one expression of the form, $\tilde{v}_{fF(e)} = \tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)}$ for $\tilde{u}^i_{f_iF(e_i)} \in \tilde{V}_{fF(e)}$.
Then $\tilde{v}_{fF(e)}$ is called fuzzy soft internal direct sum of fuzzy soft subspaces $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \dots, \tilde{u}^n_{f_nF(e_n)}$.

Definition 3.14: (Fuzzy soft external direct sum)
Let $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \dots, \tilde{v}^n_{f_nF(e_n)}$ be fuzzy soft vector space over a field F and $(v_{fF(e)}, A)$ be a fuzzy soft space having fuzzy soft n- ordered tuples $(\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \dots, \tilde{v}^n_{f_nF(e_n)}) \in \tilde{V}$.

Then $\tilde{v}_{fF(e)}$ is called fuzzy soft external direct sum if,

i. Two fuzzy soft n-tuples $(\tilde{v}^1_{f_1F(e_1)}, \dots, \tilde{v}^n_{f_nF(e_n)})$ and $(\tilde{v}^{1'}_{f_1F(e_1)}, \dots, \tilde{v}^{n'}_{f_nF(e_n)})$ are equal, if and only if $\tilde{v}^i_{f_iF(e_i)} = \tilde{v}^{i'}_{f_iF(e_i)}$.

ii. $[\tilde{v}^1_{f_1F(e_1)}, \dots, \tilde{v}^n_{f_nF(e_n)}] + [\tilde{v}^{1'}_{f_1F(e_1)}, \dots, \tilde{v}^{n'}_{f_nF(e_n)}] = (\widetilde{v^1 + v^{1'}})_{f_1F(e_1)}, \dots, (\widetilde{v^n + v^{n'}})_{f_nF(e_n)}$.

iii. $\tilde{\alpha}[\tilde{v}^1_{f_1F(e_1)}, \dots, \tilde{v}^n{}_{f_nF(e_n)}] = (\widetilde{\alpha v^1})_{f_1F(\alpha e_1)}, \dots, (\widetilde{\alpha v^n})_{f_nF(\alpha e_n)}$.

Fuzzy soft external direct sum is denoted by $\tilde{v}_{fF(e)} = \tilde{v}^1_{f_1F(e_1)} \oplus \tilde{v}^2_{f_2F(e_2)} \oplus \dots \oplus \tilde{v}^n_{f_nF(e_n)}$.

IV. FUNDAMENTAL RESULTS BASED ON FUZZY SOFT VECTOR SPACES

Properties 4.1:
If $(v_{fF(e)}, A)$ is a fuzzy soft vector space over F, then

1) $\tilde{\alpha}0 = 0$ for $\tilde{\alpha} \in F$
2) $0(\tilde{v}_{fF(e)}) = 0$ for $\tilde{v}_{fF(e)} \in \tilde{V}$
3) $(-\tilde{\alpha})\tilde{v}_{fF(e)} = -(\widetilde{\alpha v})_{fF(\alpha e)}$ for $\tilde{\alpha} \in F$, $\tilde{v}_{fF(e)} \in \tilde{V}$
4) If $\tilde{v}_{fF(e)} \neq 0$, then $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$ implies that $\tilde{\alpha} = 0$
5) If $\tilde{\alpha} \neq 0$, then $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$ implies that $\tilde{v}_{fF(e)} = 0$
6) $\tilde{\alpha}[\tilde{v}_{fF(e)} - \tilde{w}_{fF(e)}] = (\widetilde{\alpha v})_{fF(\alpha e)} - (\widetilde{\alpha w})_{fF(\alpha e)}$

Proof:

1) $\tilde{\alpha}0 = \tilde{\alpha}0 + 0$

$\tilde{\alpha}(0 + 0) = \tilde{\alpha}0 + 0$

$\tilde{\alpha}0 + \tilde{\alpha}0 = \tilde{\alpha}0 + 0$

$\tilde{\alpha}0 = 0$ (By left cancellation law)

2) $0\,(\tilde{v}_{fF(e)}) = 0(\tilde{v}_{fF(e)}) + 0$
$(0 + 0)\,(\tilde{v}_{fF(e)}) = 0(\tilde{v}_{fF(e)}) + 0$
$0(\tilde{v}_{fF(e)}) + 0(\tilde{v}_{fF(e)}) = 0(\tilde{v}_{fF(e)}) + 0$
$0(\tilde{v}_{fF(e)}) = 0$ (By left cancellation law)

3) We know that, $0(\tilde{v}_{fF(e)}) = 0$
$(\tilde{\alpha} + (-\tilde{\alpha}))\tilde{v}_{fF(e)} = 0$
$(\widetilde{\alpha v})_{fF(\alpha e)} + (-\widetilde{\alpha v})_{fF(\alpha e)} = 0$
$(-\widetilde{\alpha v})_{fF(\alpha e)} = -[(\widetilde{\alpha v})_{fF(\alpha e)}]$

4) If $\tilde{v}_{fF(e)} \neq 0$, then $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$

To prove: $\tilde{\alpha} = 0$
Since $\tilde{v}_{fF(e)} \neq 0$ and $\tilde{v}_{fF(e)} \in \tilde{V}$

then $\tilde{v}^{-1}{}_{fF(e)}$ exist in $\tilde{V}$

Now $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$

$(\widetilde{\alpha v})_{fF(\alpha e)} \cdot (\tilde{v}^{-1}{}_{fF(e)}) = 0 \cdot (\tilde{v}^{-1}{}_{fF(e)})$

$(\tilde{\alpha})[(\tilde{v}_{fF(e)}).\ (\tilde{v}^{-1}{}_{fF(e)})] = 0$

$\therefore \tilde{\alpha} = 0$

5) If $\tilde{\alpha} \neq 0$, then $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$

To prove: $\tilde{V}_{fF(e)} = 0$

Since $\tilde{\alpha} \neq 0$ and $\tilde{\alpha} \in F$ then $\tilde{\alpha}^{-1}$ exist in F

Now $(\widetilde{\alpha v})_{fF(\alpha e)} = 0$

$(\tilde{\alpha}^{-1}).(\widetilde{\alpha v})_{fF(\alpha e)} = \tilde{\alpha}^{-1} . 0$

$(\tilde{\alpha}^{-1}.\tilde{\alpha})\tilde{v}_{fF(e)} = 0$

$\therefore \tilde{v}_{fF(e)} = 0$

6) $\tilde{\alpha}[\tilde{v}_{fF(e)} - \widetilde{w}_{fF(e)}] = \tilde{\alpha}[\tilde{v}_{fF(e)} + (-\widetilde{w}_{fF(e)})]$

$= (\widetilde{\alpha v})_{fF(\alpha e)} + (-(\widetilde{\alpha w})_{fF(\alpha e)})$

$= (\widetilde{\alpha v})_{fF(\alpha e)} - (\widetilde{\alpha w})_{fF(\alpha e)}$

Lemma 4.2:

Let $(v_{fF(e)}, A)$ be a fuzzy soft vector space and $\widetilde{w}_{fF(e)}$ a subspace of $\tilde{v}_{fF(e)}$, then $\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}$ along with the operation

1) $[\tilde{v}^1_{f_1F(e_1)} + \widetilde{w}_{fF(e)}] + [\tilde{v}^2_{f_2F(e_2)} + \widetilde{w}_{fF(e)}] = (\widetilde{v^1 + v^2})_{f_1F(e_1) + f_2F(e_2)} + \widetilde{w}_{fF(e)}$

$\tilde{\alpha}[\tilde{v}^1_{f_1F(e_1)} + \widetilde{w}_{fF(e)}] = (\widetilde{\alpha v^1})_{fF(\alpha e)} + \widetilde{w}_{fF(e)}$ is a fuzzy soft vector space.

Property 4.3: Every $ker\, \tilde{T}$(fuzzy soft kernel of homomorphism) is a fuzzy soft subspace of $\tilde{v}_{fF(e)}$.

Property 4.4: The intersection of two fuzzy soft subspace of $\tilde{v}_{fF(e)}$ is a fuzzy soft subspace of $\tilde{v}_{fF(e)}$.

Property 4.5: Let $(v_{fF(e)}, A)$ be a fuzzy soft vector space and $\widetilde{w}^i_{f_iF(e_i)}$ a family of fuzzy soft subspace of $\tilde{v}_{fF(e)}$. Then $\cap\, \widetilde{w}^i_{f_iF(e_i)}$ is also a fuzzy soft subspace of $\tilde{v}_{fF(e)}$.

Property 4.6: The union of two fuzzy soft subspace of a fuzzy soft vector space need not be a fuzzy soft subspace.

Theorem 4.7:

If $(v_{fF(e)}, A)$ is the fuzzy soft internal direct sum of $(\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$ then $\tilde{v}_{fF(e)}$ is fuzzy soft isomorphic to the fuzzy soft external direct sum of $(\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$.

Proof:

Let $\tilde{u}_{fF(e)} \in \tilde{V}$

Given $\tilde{v}_{fF(e)}$ is the fuzzy soft internal direct sum of $(\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$.

(i.e.,) $\tilde{v}_{fF(e)} = (\tilde{u}^1_{f_1F(e_1)}) + (\tilde{u}^2_{f_2F(e_2)}) + \cdots + (\tilde{u}^n_{f_nF(e_n)})$

Define a mapping

$\tilde{T}: \tilde{v}_{fF(e)} \to \tilde{u}^1_{f_1F(e_1)} \oplus \tilde{u}^2_{f_2F(e_2)} \oplus \dots \oplus \tilde{u}^n_{f_nF(e_n)}$ by,

$\tilde{T}(\tilde{v}_{fF(e)}) = \tilde{T}[\tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)}]$

$= (\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$

(1) $\tilde{T}$ is well defined as $\tilde{v}_{fF(e)} \in \tilde{V}$:

$\tilde{v}_{fF(e)} = (\tilde{u}^1_{f_1F(e_1)}) + (\tilde{u}^2_{f_2F(e_2)}) + \cdots + (\tilde{u}^n_{f_nF(e_n)})$ has one and only representation.

(2) $\tilde{T}$ is onto:

$\tilde{T}(\tilde{v}_{fF(e)}) = (\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$

$\Longrightarrow (\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)})$ is an image of $\tilde{v}_{fF(e)}$.

$\Longrightarrow (\tilde{u}^1_{f_1F(e_1)}), (\tilde{u}^2_{f_2F(e_2)}), \dots, (\tilde{u}^n_{f_nF(e_n)}) \in \tilde{u}^1_{f_1F(e_1)} \oplus \tilde{u}^2_{f_2F(e_2)} \oplus \dots \oplus \tilde{u}^n_{f_nF(e_n)}$ is an image of $\tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)} \in \tilde{V}$.

(3) $\tilde{T}$ is one to one:

Let $\tilde{T}(\tilde{v}_{fF(e)}) = \tilde{T}(\widetilde{w}_{fF(e)})$

$\Longrightarrow \tilde{T}[\tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)}] = \tilde{T}[\widetilde{w}^1_{f_1F(e_1)} + \widetilde{w}^2_{f_2F(e_2)} + \cdots + \widetilde{w}^n_{f_nF(e_n)}]$

where $\tilde{u}^i_{f_iF(e_i)}, \widetilde{w}^i_{f_iF(e_i)} \in \tilde{V}$

$\Longrightarrow \tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \dots, \tilde{u}^n_{f_nF(e_n)} = \widetilde{w}^1_{f_1F(e_1)}, \widetilde{w}^2_{f_2F(e_2)}, \dots, \widetilde{w}^n_{f_nF(e_n)}$

$\Longrightarrow \tilde{u}^1_{f_1F(e_1)} = \widetilde{w}^1_{f_1F(e_1)}$, $\tilde{u}^2_{f_2F(e_2)} = \widetilde{w}^2_{f_2F(e_2)}, \dots, \tilde{u}^n_{f_nF(e_n)} = \widetilde{w}^n_{f_nF(e_n)}$.

$\Longrightarrow \tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)} = \widetilde{w}^1_{f_1F(e_1)} + \widetilde{w}^2_{f_2F(e_2)} + \cdots + \widetilde{w}^n_{f_nF(e_n)}$

$\Longrightarrow \tilde{v}_{fF(e)} = \widetilde{w}_{fF(e)}$

(4) $\tilde{T}$ is homomorphism:

i) $\tilde{T}[\tilde{v}_{fF(e)} + \widetilde{w}_{fF(e)}] = \tilde{T}[\tilde{u}^1_{f_1F(e_1)} + \cdots + \tilde{u}^n_{f_nF(e_n)} + \widetilde{w}^1_{f_1F(e_1)} + \cdots + \widetilde{w}^n_{f_nF(e_n)}]$

$= \tilde{T}[(\tilde{u}^1_{f_1F(e_1)} + \widetilde{w}^1_{f_1F(e_1)}) + \cdots + (\tilde{u}^n_{f_nF(e_n)} + \widetilde{w}^n_{f_nF(e_n)})]$

$= (\tilde{u}^1_{f_1F(e_1)} + \widetilde{w}^1_{f_1F(e_1)}), \dots, (\tilde{u}^n_{f_nF(e_n)} + \widetilde{w}^n_{f_nF(e_n)})$

$= (\tilde{u}^1_{f_1F(e_1)}, \dots, \tilde{u}^n_{f_nF(e_n)}) + (\widetilde{w}^1_{f_1F(e_1)} + \widetilde{w}^n_{f_nF(e_n)})$

$\tilde{T}[\tilde{v}_{fF(e)} + \widetilde{w}_{fF(e)}] = \tilde{T}(\tilde{v}_{fF(e)}) + \tilde{T}(\widetilde{w}_{fF(e)})$

ii) $\tilde{T}[(\widetilde{\alpha v})_{fF(\alpha e)}] = \tilde{T}[\tilde{\alpha}(\tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)})]$

$= \tilde{\alpha}\tilde{T}[\tilde{u}^1_{f_1F(e_1)} + \tilde{u}^2_{f_2F(e_2)} + \cdots + \tilde{u}^n_{f_nF(e_n)}]$

$= \tilde{\alpha}[\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^n_{f_nF(e_n)}]$

$\tilde{T}[(\widetilde{\alpha v})_{fF(\alpha e)}] = \tilde{\alpha}[\tilde{T}(\tilde{v}_{fF(e)})]$

$\Longrightarrow \tilde{T}$ is fuzzy soft homomorphism

$\Longrightarrow \tilde{T}$ is fuzzy soft isomorphism

$\Longrightarrow \tilde{v}_{fF(e)} = \tilde{u}^1_{f_1F(e_1)} \oplus \tilde{u}^2_{f_2F(e_2)} \oplus \ldots \oplus \tilde{u}^n_{f_nF(e_n)}$

$\therefore$ $\tilde{v}_{fF(e)}$ is fuzzy soft isomorphic to the fuzzy soft external direct sum of $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^n_{f_nF(e_n)}$.

Lemma 4.8:

L($\tilde{S}_{fF(e)}$) is a fuzzy soft subspace of $\tilde{v}_{fF(e)}$.

Lemma 4.9:

If $\tilde{S}_{fF(e)}$, $\tilde{T}_{fF(e)}$ are fuzzy soft subspaces of $\tilde{V}_{fF(e)}$, then

1)$\tilde{S}_{fF(e)} \subset \tilde{T}_{fF(e)} \Rightarrow L(\tilde{S}_{fF(e)}) \subset L(\tilde{T}_{fF(e)})$.

2) $L[\tilde{S}_{fF(e)} \cup \tilde{T}_{fF(e)}] = L(\tilde{S}_{fF(e)}) + L(\tilde{T}_{fF(e)})$

3) $L[L(\tilde{S}_{fF(e)})] = L(\tilde{S}_{fF(e)})$

Lemma 4.10:

If $(\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^n_{f_nF(e_n)}) \in \tilde{V}$ are fuzzy soft linearly independent, then every element in their fuzzy soft linear span has a unique representation in the form $(\widetilde{\lambda_1 v^1})_{f_1F(\lambda e_1)} + \cdots + (\widetilde{\lambda_n v^n})_{f_nF(\lambda e_n)}$ with the $\tilde{\lambda}_i \in F$.

Theorem 4.11:

If $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^n_{f_nF(e_n)}$ are in $\tilde{V}$ then either they are fuzzy soft linearly independent or some $\tilde{v}^k_{f_kF(e_k)}$ is a fuzzy soft linear combination of the proceeding ones $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^{k-1}_{f_{k-1}F(e_{k-1})}$.

Proof:

If $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^n_{f_nF(e_n)}$ are fuzzy soft linearly independent then there is nothing to prove.

Assume that they are fuzzy soft linearly independent.

Hence we can find scalars $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n-1}$ not all zero. Such that,

$$(\widetilde{\alpha_1 v^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_n v^n})_{f_nF(\alpha_n e_n)} = 0 \ldots.. \quad \mathbf{(1)}$$

Let k be the largest integer, such that$\tilde{\alpha}_k \neq 0$.

Since $\tilde{\alpha}_i = 0$ for i>k, i = k+1,…,n

$$\tilde{\alpha}_{k+1} = \tilde{\alpha}_{k+2} = \cdots = \tilde{\alpha}_n = 0$$

$$\mathbf{(1)} \Longrightarrow (\widetilde{\alpha_1 v^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_k v^k})_{f_kF(\alpha_k e_k)} = 0$$

$$-(\widetilde{\alpha_k v^k})_{f_kF(\alpha_k e_k)} = (\widetilde{\alpha_1 v^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_{k-1} v^{k-1}})_{f_{k-1}F(\alpha_{k-1} e_{k-1})}$$

$(-\tilde{\alpha}_k^{-1})(\widetilde{-\alpha_k v^k})_{f_kF(\alpha_k e_k)} = (-\tilde{\alpha}_k^{-1})[(\widetilde{\alpha_1 v^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_{k-1} v^{k-1}})_{f_{k-1}F(\alpha_{k-1} e_{k-1})}]$

$$\tilde{v}^k_{f_kF(e_k)} = (-\tilde{\alpha}_k^{-1}\tilde{\alpha}_1)\tilde{v}^1_{f_1F(e_1)} + \cdots + (-\tilde{\alpha}_k^{-1}\tilde{\alpha}_{k-1})\tilde{v}^{k-1}_{f_{k-1}F(e_{k-1})}$$

Thus $\tilde{v}^k_{f_kF(e_k)}$ is a fuzzy soft linear combination of the proceeding vectors

$(\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^{k-1}_{f_{k-1}F(e_{k-1})}$.

Theorem 4.12:

Any fuzzy soft finite dimensional vector space contains a fuzzy soft basis

Proof:

Let $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^m_{f_mF(e_m)}$ be a fuzzy soft spanning set of $\tilde{v}_{fF(e)}$.

If $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^m_{f_mF(e_m)}$ is fuzzy soft linearly independent then form a fuzzy soft basis and there is nothing to prove.

Assume $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^m_{f_mF(e_m)}$ is fuzzy soft linearly dependent then one of the fuzzy soft vectors say $\tilde{u}^k_{f_kF(e_k)}$ is a fuzzy soft linear combination of the remaining $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^{k-1}_{f_{k-1}F(e_{k-1})}$ removing these vectors and obtaining a set of (k-1) fuzzy soft vectors. If this set $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^{k-1}_{f_{k-1}F(e_{k-1})}$is fuzzy soft linearly independent, then form a fuzzy soft basis but if $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^{k-1}_{f_{k-1}F(e_{k-1})}$ is fuzzy soft linearly dependent.

Then the above process is continued in this way, we can get a fuzzy soft linear independent spanning set and hence a fuzzy soft basis.

Thus the fuzzy soft subset of $\tilde{u}^1_{f_1F(e_1)}, \tilde{u}^2_{f_2F(e_2)}, \ldots, \tilde{u}^m_{f_mF(e_m)}$ forms a fuzzy soft basis of $\tilde{v}_{fF(e)}$.

Lemma 4.13:

If $\tilde{v}^1_{f_1F(e_1)}, \tilde{v}^2_{f_2F(e_2)}, \ldots, \tilde{v}^n_{f_nF(e_n)}$ is a fuzzy soft basis of $\tilde{v}_{fF(e)}$ over F and if $\widetilde{w}^1_{f_1F(e_1)}, \widetilde{w}^2_{f_2F(e_2)}, \ldots, \widetilde{w}^m_{f_mF(e_m)}$in $\tilde{v}_{fF(e)}$ are fuzzy soft linearly independent over F, then $m \leq n$.

Lemma 4.14:

If $\tilde{v}_{fF(e)}$ is fuzzy soft finite finite-dimensional over F, then any two fuzzy soft basis of $\tilde{v}_{fF(e)}$ have the same number of elements.

Lemma 4.15:
Any two fuzzy soft finite-dimensional vector space over F of the same dimension are fuzzy soft isomorphic.

Lemma 4.16:
If $\tilde{v}_{fF(e)}$ is fuzzy soft finite dimensional and if $\widetilde{w}_{fF(e)}$ is a fuzzy soft subspace of $\tilde{v}_{fF(e)}$, then $\widetilde{w}_{fF(e)}$ is fuzzy soft finite dimensional, $\dim \widetilde{W}_{fF(e)} \leq \dim \tilde{V}_{fF(e)}$ and $\dim(\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}) = \dim \tilde{v}_{fF(e)} - \dim \widetilde{w}_{fF(e)}$.

Theorem 4.17:
If $\tilde{v}_{fF(e)}$ is fuzzy soft finite dimensional and if $\widetilde{w}_{fF(e)}$ is a fuzzy soft subspace of $\tilde{v}_{fF(e)}$, then $\widetilde{w}_{fF(e)}$ is fuzzy soft finite dimensional, $\dim \widetilde{W}_{fF(e)} \leq \dim \tilde{V}_{fF(e)}$ and $\dim(\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}) = \dim \tilde{v}_{fF(e)} - \dim \widetilde{w}_{fF(e)}$.

Proof:
By lemma 4.5,
If $n = \dim \tilde{v}_{fF(e)}$
Then any $n+1$ elements in $\tilde{v}_{fF(e)}$ are fuzzy soft linearly dependent.
In particular, any $n+1$ elements in $\widetilde{w}_{fF(e)}$ are fuzzy soft linearly dependent.
Thus we can find a largest set of fuzzy soft linearly elements in $\widetilde{w}_{fF(e)}, \widetilde{w}^1_{f_1F(e_1)}, \dots, \widetilde{w}^m_{f_mF(e_m)}$ and $m \leq n$.
If $\widetilde{w}_{fF(e)} \in \widetilde{W}$, then $\widetilde{w}^1_{f_1F(e_1)}, \dots, \widetilde{w}^m_{f_mF(e_m)}$ is a fuzzy soft linearly dependent sets, when $(\widetilde{\alpha w})_{fF(\alpha e)} + (\widetilde{\alpha_1 w^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_m w^m})_{f_mF(\alpha_m e_m)} = 0$not all of the $\tilde{\alpha}_i = 0$.
If $\alpha = 0$ by the fuzzy soft linear independence of the $\widetilde{w}^i_{f_iF(e_i)}$ we would get that each $\tilde{\alpha}_i = 0$, a contradiction. Thus $\tilde{\alpha} \neq 0$
So, $\widetilde{w}_{fF(e)} = -\tilde{\alpha}^{-1}[(\widetilde{\alpha_1 w^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_m w^m})_{f_mF(\alpha_m e_m)}]$
Consequently, $(\widetilde{\alpha_1 w^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_m w^m})_{f_mF(\alpha_m e_m)}$ fuzzy soft span $\widetilde{w}_{fF(e)}$.
By this, $\widetilde{w}_{fF(e)}$ is fuzzy soft finite dimensional over F.
It has a fuzzy soft basis of m elements, where $m \leq n$ from the definition of fuzzy soft dimension if then follows that $\dim \widetilde{w}_{fF(e)} \leq \dim \tilde{v}_{fF(e)}$.

To prove:
$$\dim(\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}) = \dim \tilde{v}_{fF(e)} - \dim \widetilde{w}_{fF(e)}$$
Let $\tilde{v}_{fF(e)}$ is fuzzy soft finite dimensional
Let $n = \dim \tilde{v}_{fF(e)}$
By the theorem, "$\widetilde{w}_{fF(e)}$ is fuzzy soft finite dimensional and $\dim \widetilde{w}_{fF(e)} \leq \dim \tilde{v}_{fF(e)}$".
Let $\widetilde{w}^1_{f_1F(e_1)}, \dots, \widetilde{w}^m_{f_mF(e_m)}$be a fuzzy soft basis of $\widetilde{w}_{fF(e)}$. This is a fuzzy soft linearly independent set in $\tilde{v}_{fF(e)}$.
Hence it can be extended to form a fuzzy soft basis of $\tilde{v}_{fF(e)}$.
$\Longrightarrow \{\widetilde{w}^1_{f_1F(e_1)}, \dots, \widetilde{w}^m_{f_mF(e_m)}, \tilde{v}^1_{f_1F(e_1)}, \dots, \tilde{v}^r_{f_rF(e_r)}\}$be a fuzzy soft basis of $\tilde{v}_{fF(e)}$.
$\dim \tilde{v}_{fF(e)} = m + r$ and $\dim \widetilde{w}_{fF(e)} = m$
$\Longrightarrow n = \dim \tilde{v}_{fF(e)} = m + r$
$\Longrightarrow n = m + r$
$\Longrightarrow (n - m) = r$
Denote any element in $\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}$as $\bar{\tilde{v}}_{fF(e)} = \tilde{v}_{fF(e)} + \widetilde{w}_{fF(e)}, \tilde{v}_{fF(e)} \in \tilde{V}$, $\bar{\tilde{v}}_{fF(e)}$is image of $\tilde{v}_{fF(e)}$.
Since, as$\tilde{v}_{fF(e)} \in \tilde{V}$ is of form
$$\tilde{v}_{fF(e)} = (\widetilde{\alpha_1 w^1})_{f_1F(\alpha_1 e_1)} + \cdots + (\widetilde{\alpha_m w^m})_{f_mF(\alpha_m e_m)} + (\widetilde{\beta_1 v^1})_{f_1F(\beta_1 e_1)} + \cdots + (\widetilde{\beta_r v^r})_{f_rF(\beta_r e_r)}$$
where $\tilde{\alpha}_i \& \tilde{\beta}_i \in F$
and $\bar{\tilde{v}}_{fF(e)} = (\tilde{\beta}_1 \bar{\tilde{v}}^1)_{f_1F(\beta_1 e_1)} + \cdots + (\tilde{\beta}_r \bar{\tilde{v}}^r)_{f_rF(\beta_r e_r)}$
$\bar{\tilde{v}}^1{}_{f_1F(e_1)}, \bar{\tilde{v}}^2{}_{f_2F(e_2)}, \dots, \bar{\tilde{v}}^r{}_{f_rF(e_r)}$ are fuzzy soft basis of $\tilde{V}$.
Thus, $\bar{\tilde{v}}^1{}_{f_1F(e_1)}, \bar{\tilde{v}}^2{}_{f_2F(e_2)}, \dots, \bar{\tilde{v}}^r{}_{f_rF(e_r)}$ fuzzy soft span $\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}$.

We claim that they are fuzzy soft linearly independent:

For if $(\tilde{\gamma}_1 \bar{\tilde{v}}^1)_{f_1F(\gamma_1 e_1)} + \cdots + (\tilde{\gamma}_r \bar{\tilde{v}}^r)_{f_rF(\gamma_r e_r)} = \bar{0}$
$$\Longrightarrow \tilde{\gamma}_1(\tilde{v}^1_{f_1F(e_1)} + \widetilde{w}_{fF(e)}) + \cdots + \tilde{\gamma}_r(\tilde{v}^r_{f_rF(e_r)} + \widetilde{w}_{fF(e)}) = 0 + \widetilde{w}_{fF(e)}$$
$$\Longrightarrow [(\widetilde{\gamma_1 v^1})_{f_1F(\gamma_1 e_1)} + \widetilde{w}_{fF(e)}] + \cdots + [(\widetilde{\gamma_r v^r})_{f_rF(\gamma_r e_r)} + \widetilde{w}_{fF(e)}] = \widetilde{w}_{fF(e)}$$
$$\Longrightarrow [(\widetilde{\gamma_1 v^1})_{f_1F(\gamma_1 e_1)} + \cdots + (\widetilde{\gamma_r v^r})_{f_rF(\gamma_r e_r)}] + \widetilde{w}_{fF(e)} = \widetilde{w}_{fF(e)}$$
$$\Longrightarrow (\widetilde{\gamma_1 v^1})_{f_1F(\gamma_1 e_1)} + \cdots + (\widetilde{\gamma_r v^r})_{f_rF(\gamma_r e_r)} \in \widetilde{W}$$
So that,
$$(\widetilde{\gamma_1 v^1})_{f_1F(\gamma_1 e_1)} + \cdots + (\widetilde{\gamma_r v^r})_{f_rF(\gamma_r e_r)} = (\widetilde{\lambda_1 w^1})_{f_1F(\lambda_1 e_1)} + \cdots + (\widetilde{\lambda_m w^m})_{f_mF(\lambda_m e_m)}$$
$\because \{\widetilde{w}^1_{f_1F(e_1)}, \dots, \widetilde{w}^m_{f_mF(e_m)}$fuzzy soft linearly independent}
$\Longrightarrow \tilde{\gamma}_1 = \cdots = \tilde{\gamma}_r = \tilde{\lambda}_1 \dots = \tilde{\lambda}_m$
Hence we have show that $\tilde{v}_{fF(e)}/\widetilde{w}_{fF(e)}$ has a fuzzy soft basis of r elements namely

$\tilde{\bar{v}}^1{}_{f_1F(e_1)}, \tilde{\bar{v}}^2{}_{f_2F(e_2)}, \dots, \tilde{\bar{v}}^r{}_{f_rF(e_r)}$.

$$\Longrightarrow \dim(\tilde{v}_{fF(e)}/\,\widetilde{w}_{fF(e)}) = r = n - m$$

$$\Longrightarrow \dim(\tilde{v}_{fF(e)}/\,\widetilde{w}_{fF(e)}) = \dim \tilde{v}_{fF(e)} - \dim \widetilde{w}_{fF(e)}$$

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