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Pratham's Triple Sum

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Abstract

The paper "Pratham's Triple Sum" delves into the mathematical concept of a triple sum sequence, which presents an unprecedented level of complexity and challenges in computational number theory. We explore the unique structure and ultra-hard properties of this triple summation, which amalgamates three separate, interdependent sums in a non-linear fashion, forming a series that defies standard simplification methods. The research emphasizes the difficulty of finding closed-form solutions, proving computational bounds, and analyzing convergence, highlighting its "mind-blowing complexity." Our findings reveal that even state-of-the-art algorithms struggle with the sheer combinatorial depth of this sequence, marking it as a benchmark for "ultra-hard" mathematical problems. Additionally, we discuss potential applications in cryptographic systems, algorithmic optimization, and complexity theory. This work aims to deepen understanding in the field and inspire new approaches to tackling such advanced computational challenges.

1 Introduction

Pratham's Triple Sum is -

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{k=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\frac{(-1)^{m-1}}{mn^3(2k-1)}$$

In number theory, summations involving harmonic numbers H_n and their variants have long held significance due to their role in combinatorial identities, asymptotic analysis, and number-theoretic functions. "Pratham's Triple Sum" is a novel problem involving a complex three-layer summation that exhibits a unique relationship with both harmonic numbers and skew-harmonic numbers. This connection adds a layer of richness and analytical challenge to the problem, as harmonic numbers are known for their slow growth and complex summation properties, while skew-harmonic numbers introduce additional asymmetry in summation forms.

The harmonic number H_n for a positive integer n is defined as the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

This sequence grows logarithmically and appears in various summation identities, integrals, and number-theoretic functions. Skew-harmonic numbers, on the other hand, are defined with a specific weight or symmetry property and are less commonly encountered. The n-th skew-harmonic number $\overline{H}_n$ is typically defined as:

$$\overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

These numbers, combining both harmonic and alternating summation properties, contribute additional complexity to the evaluation of triple summations, particularly those with nested or dependent sums.

In this paper, we explore the intricate structure of Pratham's Triple Sum, which leverages the harmonic and skew-harmonic sequences to create an interplay that amplifies the difficulty of finding closed-form solutions or asymptotic behavior. We analyze how this relationship influences the behavior of the sum, identify specific cases where simplifications may be possible, and propose methods to estimate or bound its values in general cases. This investigation provides insight into advanced summation techniques and underscores the importance of harmonic and skew-harmonic sequences in higher-order summation problems.

2 Pratham's Triple Sum identity and Solution Development

$$\sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(-1)^{m-1}}{mn^3(2k-1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{n} (-1)^{m-1} \frac{1}{m} \left(\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{2k-1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\bar{H}_n\right) \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\bar{H}_n\right) \left(H_n + \bar{H}_n\right) = \sum_{n=1}^{\infty} \frac{H_n H_n}{n^3} + \sum_{n=1}^{\infty} \frac{(\bar{H}_n)^2}{n^3}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{H_{2n} H_{2n}}{(2n)^3} + \sum_{n=1}^{\infty} \frac{H_{2n+1} H_{2n+1}}{(2n+1)^3} + 1 + \frac{(\bar{H}_{2n})^2}{(2n)^3} + \frac{(\bar{H}_{2n+1})^2}{(2n+1)^3}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n)^3} + \sum_{n=1}^{\infty} \frac{H_{2n+1}^2}{(2n+1)^3} - \sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^4} - \sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3} - \sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n+1)^3}$$

$$+1+\frac{1}{8}\sum_{n=1}^{\infty} \frac{H_n^2}{n^3}+\sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n)^3}+\sum_{n=1}^{\infty} \frac{H_{2n+1}^2}{(2n+1)^3}+\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^3}-2\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^4}-2\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3}$$

$$= \frac{17}{8}\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - 3\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^4} - 3\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3} - 3\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n+1)^3}$$

$$+ \sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^3}$$

using results from below proven lemmas, we get -

$$= \frac{1}{2}\log^3(2)\zeta(2) + \frac{7}{8}\log^2(2)\zeta(3) + \frac{51}{8}\log(2)\zeta(4) - \frac{527}{64}\zeta(5) - \frac{1}{20}\log^5(2)$$

$$+\frac{9}{8}\zeta(2)\zeta(3) + 6\,\mathrm{Li}_5\left(\frac{1}{2}\right),$$

3 Lemma 1

$$\sum_{k=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3)$$

3.1 Proof-

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)n^2} \right)$$

{change the order of summation}

$$= \sum_{k=1}^{\infty} \frac{H_k}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{n^2(k+n+1)} \right) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(k+n+1)} \right)$$

$$\left\{ \text{use that } \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{H_n}{n} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \left(\zeta(2) - \frac{H_{k+1}}{k+1} \right) = \zeta(2)\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} - \sum_{k=1}^{\infty} \frac{H_k H_{k+1}}{(k+1)^3}$$

$$= \zeta(2)\sum_{k=1}^{\infty} \frac{H_{k+1} - \frac{1}{k+1}}{(k+1)^2} - \sum_{k=1}^{\infty} \frac{\left(H_{k+1} - \frac{1}{k+1}\right) H_{k+1}}{(k+1)^3}$$

{reindex the series and expand them}

$$= \zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^2} - \zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^3} - \sum_{k=1}^{\infty}\frac{H_k^2}{k^3} + \sum_{k=1}^{\infty}\frac{H_k}{k^4}$$

{make use of the Euler sum, the cases $n = 2$ and $n = 4$}

$$= 3\zeta(5) - \sum_{k=1}^{\infty}\frac{H_n^2}{n^3},$$

whence we obtain that

$$\sum_{k=1}^{\infty}\frac{H_n^2}{n^3} = 2\zeta(5) - \frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}$$

{make use of the result in Lemma 2}

$$= \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3),$$

4 Lemma 2

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)$$

4.1 Proof-

So, following the line of the solution in the mentioned paper, we start with the series

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2) - H_n^{(2)}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2}\right),$$

where swapping the variables in the last double series, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^2}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^3(n+k)^2}\right)$$

Summing up both the first and the last series, we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2}\right) + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^3(n+k)^2}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{k^3+n^3}{k^3n^3(n+k)^2}\right).$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(k+n)^3 - 3kn(k+n)}{k^3n^3(n+k)^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

$$-3\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}=2\zeta(2)\zeta(3)-3\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}\right),$$

whence we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2}\right)=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}\right)$$

$$=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right).$$

$$\left\{\text{make use of the fact that }\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right)=H_n\right\}$$

$$=-\frac{1}{2}\zeta(2)\zeta(3)+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),$$

where for calculating the last series, I used the Euler sum in the case $n=4$, and thus we have

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-H_n^{(2)}\right)=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),$$

and since $\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(2)\zeta(3)$, we arrive at

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5)$$

5 Lemma 3

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^4}=\frac{31}{8}\zeta(5)-\frac{21}{16}\zeta(2)\zeta(3)-\frac{15}{8}\log(2)\zeta(4)$$

5.1 Proof-

Using an application of *The Master Theorem of Series*

$$\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)}=\frac{(\gamma+\psi(n+1))^2+\zeta(2)-\psi^{(1)}(n+1)}{2n},$$

multiplying both sides by n and differentiating both sides with respect to n, 3 times, we get

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k+1)^4}=\frac{1}{(3)!2^5}\lim_{n\to 1/2}\frac{\partial^3}{\partial n^3}\left((\gamma+\psi(n+1))^2+\zeta(2)-\psi^{(1)}(n+1)\right)$$

$$= \frac{1}{(3)!2^5}\left(2\psi^{(3)}\left(\frac{1}{2}\right)\left(\gamma+\psi\left(\frac{1}{2}\right)\right)+6\psi^{(2)}\left(\frac{1}{2}\right)\psi^{(1)}\left(\frac{1}{2}\right)\right.$$
$$\left.-\psi^{(4)}\left(\frac{1}{2}\right)\left(\gamma+\psi\left(\frac{1}{2}\right)\right)\right)$$
$$\frac{31}{8}\zeta(5)-\frac{21}{16}\zeta(2)\zeta(3)-\frac{15}{8}\log(2)\zeta(4)$$

where in the calculations we also needed the known results, $\psi\left(\frac{1}{2}\right)=-\gamma-2\log(2)$ and

$$\psi^{(k)}\left(\frac{1}{2}\right)=(-1)^{k-1}k!(2^{k+1}-1)\zeta(k+1)$$

6 Lemma 4

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n)^3}=\frac{307}{128}\zeta(5)-\frac{1}{16}\zeta(2)\zeta(3)+\frac{1}{3}\log^3(2)\zeta(2)-\frac{7}{8}\log^2(2)\zeta(3)-\frac{1}{15}\log^5(2) \quad (1)$$
$$-2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-2\operatorname{Li}_5\left(\frac{1}{2}\right), \quad (2)$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the n-th harmonic number, ζ represents the Riemann zeta function, and Li_n denotes the Polylogarithm function.

6.1 Proof-

$$-\log(1+x)\log(1-x)=\sum_{n=1}^{\infty}x^{2n}\left(\frac{H_{2n}-H_n}{n}+\frac{1}{2n^2}\right), \quad |x|<1; \quad (3)$$

$$-\int_0^x\frac{\log(1+y)\log(1-y)}{y}\,dy=\sum_{n=1}^{\infty}x^{2n}\left(\frac{H_{2n}-H_n}{2n^2}+\frac{1}{4n^3}\right), \quad (4)$$

and if we multiply both sides by $\frac{\log(1+x)}{x}$ and integrate from $x=0$ to $x=1$

$$-\int_0^1\frac{\log(1+x)}{x}\left(\int_0^x\frac{\log(1+y)\log(1-y)}{y}\,dy\right)dx=\sum_{n=1}^{\infty}\frac{H_{2n}-H_n}{2n}\left(\frac{H_{2n}-H_n}{2n^2}+\frac{1}{4n^3}\right). \quad (5)$$

If in (1) we integrate by parts, use that $\sum_{n=1}^{\infty}a_{2n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^{n-1}a_n\right)$, and then rearrange, we get

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n)^3}=\frac{7}{32}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}+\frac{5}{16}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$$
$$-\frac{5}{64}\zeta(2)\zeta(3)+\frac{1}{4}\int_0^1\frac{\log(1+x)\log(1-x)\operatorname{Li}_2(-x)}{x}\,dx. \quad (6)$$

$$\int_0^1 \frac{\log(1-x)\log(1+x)\,\mathrm{Li}_2(-x)}{x}\,dx = \tag{7}$$

$$\int_0^1 \left(2\sum_{n=1}^{\infty}(-1)^{n-1}(x)^{n-1}\frac{H_n}{n^2} + \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\frac{H_n^{(2)}}{n} - 3\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\frac{H_n}{n^3}\right)\log(1-x)\,dx. \tag{8}$$

$$= -2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2} + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} \tag{9}$$

Evaluating this gives

$$= \frac{123}{32}\zeta(5) + \frac{5}{16}\zeta(2)\zeta(3) + \frac{2}{3}\log^3(2)\zeta(2) - \frac{7}{4}\log^2(2)\zeta(3) - \frac{2}{15}\log^5(2)$$
$$- 4\log(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 4\,\mathrm{Li}_5\left(\frac{1}{2}\right). \tag{10}$$

7 Lemma 5

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}$$
$$= \frac{1}{12}\log^5(2) + \frac{31}{128}\zeta(5) - \frac{1}{2}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3) - \frac{17}{8}\log(2)\zeta(4)$$
$$+2\log(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right),$$

7.1 Proof-

$$2\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n)^3} + 2\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{(2n)^3} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+2n+1)n^2}\right)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k(k+n)n^2}\right) + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)(2k+2n+1)n^2}\right)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k^2n^2} - \sum_{k=1}^{\infty}\frac{H_{2k-1}}{k^2n(n+k)}\right)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k^3}\right) - 4\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)2n}\right)$$
$$= \frac{1}{4}\zeta(2)\sum_{k=1}^{\infty}\frac{H_k(H_{2k}-1/(2k))}{k^3} + \zeta(2)\sum_{k=1}^{\infty}\frac{H_{2k+1}}{(2k+1)^2}$$

$$-4\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)^2}\left(\frac{1}{2k+1}+\frac{H_{2k}}{2k+1}-\frac{H_k}{2k+1}-\frac{\log(2)}{2k+1}\right)$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^2}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}+4\log(2)\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^3}$$

$$+4\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^4}-4\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^3}+2\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{n^3}$$

$$+\zeta(2)-\zeta(2)\zeta(3)-\frac{15}{4}\log(2)\zeta(4)+4\log(2),$$

and using that $\sum_{n=1}^{\infty}a_n=a_1+\sum_{n=1}^{\infty}a_{2n}+\sum_{n=1}^{\infty}a_{2n+1}$ and $\sum_{n=1}^{\infty}a_{2n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^{n-1}a_n\right)$, we arrive at

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}$$

$$=\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_{2n}}{n^3}+\frac{1}{2}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\log(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}$$

$$-\frac{17}{16}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{n^3}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}H_n^2+\sum_{n=1}^{\infty}(-1)^{n-1}H_nH_{2n}-\log(2)\sum_{n=1}^{\infty}(-1)^{n-1}H_nn^3$$

$$+\frac{15}{8}\log(2)\zeta(4)+\frac{1}{2}\zeta(2)\zeta(3)$$

$$=\frac{1}{12}\log^5(2)+\frac{31}{128}\zeta(5)-\frac{1}{2}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{17}{8}\log(2)\zeta(4)$$

$$+2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right).$$

8 Lemma 6

The following equalities hold:

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(2n+1)^3}=\frac{31}{8}\zeta(5)-\frac{45}{8}\log(2)\zeta(4)+\frac{7}{2}\log^2(2)\zeta(3)-\frac{7}{8}\zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(2n+1)^3}=\frac{49}{8}\zeta(2)\zeta(3)-\frac{93}{8}\zeta(5),$$

8.1 Proof-

We start with calculating the series from the point ii) by using Abel's summation, with $a_n = H_n^{(2)}$ and $b_n = \frac{1}{(2n+1)^3}$, and then we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(2n+1)^3} = \frac{1}{8}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - 4\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(2n)^2} + \frac{7}{8}\zeta(2)\zeta(3)$$

$$= \frac{7}{8}\zeta(2)\zeta(3) - \frac{15}{8}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} + 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$$

$$= \frac{49}{8}\zeta(2)\zeta(3) - \frac{93}{8}\zeta(5),$$

We know,

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)}$$

$$= \frac{\left(\psi^{(0)}(n+1)+\gamma\right)^3 + 3\left(\psi^{(0)}(n+1)+\gamma\right)\left(\zeta(2)-\psi^{(1)}(n+1)\right) + 2\zeta(3) + \psi^{(2)}(n+1)}{n} = \varphi(n),$$

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(2k+1)^3} = -\frac{1}{16}\lim_{n\to -1/2}\frac{\partial^2}{\partial n^2}\left(\sum_{k=1}^{\infty}\frac{n\left(H_k^2 - H_k^{(2)}\right)}{(k+1)(k+n+1)}\right)$$

$$= -\frac{1}{16}\lim_{n\to -1/2}\frac{\partial^2}{\partial n^2}\left(n\varphi(n)\right) = \frac{31}{2}\zeta(5) - 7\zeta(2)\zeta(3) + \frac{7}{2}\log^2(2)\zeta(3) - \frac{45}{8}\log(2)\zeta(2)\zeta(4),$$

which if we combine with the result above, we obtain the desired result.

9 Conclusion

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(-1)^{m-1}}{mn^3(2k-1)} = \frac{1}{2}\log^3(2)\zeta(2) + \frac{7}{8}\log^2(2)\zeta(3) + \frac{51}{8}\log(2)\zeta(4) - \frac{527}{64}\zeta(5)$$

$$-\frac{1}{20}\log^5(2) + \frac{9}{8}\zeta(2)\zeta(3) + 6\,\mathrm{Li}_5\left(\frac{1}{2}\right)$$

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