Numerical Investigation of Boundary Layer Flow across a Stretching Cylinder using Quartic Spline Method

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Abstract: In this paper, the boundary layer flow of a viscous incompressible fluid across a stretching cylinder has been considered to investigate the flow field. Because the dynamic region is nonlinear, the velocity function has been calculated numerically using the non-polynomial quartic spline method. The expression of skin friction was also obtained. Graphs have been used to analyze the velocity profile on the dimensionless parameter.

***Keywords:** Stretching cylinder, Boundary layer flow, Quartic spline, Skin friction*

1. INTRODUCTION

In fiber technology and extrusion operations, the boundary layer flow caused by stretching flat plates or cylinders is theoretically as well as practically significant and fascinating. This method is used to produce plastic films and polymer sheets. Examples include the aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, the boundary layer in condensation processes along a liquid film, the blowing of glass, the spinning of metal, the drawing of plastic films, and the extrusion of polymers. It was Sakiadis (1961) who first brought the boundary layer flow on a moving continuous solid surface into consideration. Using a stretching sheet with linearly variable surface speed, Crane (1970) expanded this idea and provided an exact solution for the steady two-dimensional flow across a stretching surface in a quiescent fluid. A similarity solution is one that, typically through a coordinate transformation, reduces the number of independent variables by at least one. The concept is similar to dimensional analysis, except the coordinates themselves are reduced into dimensionless units that scale the velocities rather than parameters, such as the Reynolds number, see F. M. White (2006).

The works of Weyl (1942), Coppel (1960), Lin and Chen (1998), Liao (1999), Partha et al. (2005), Anderson (2005), Ishak (2009), Kudenatti (2012) and Rangi et al. (2012) have discussed the boundary layer flow caused by a stretching vertical surface in a quiescent viscous and incompressible fluid when the buoyancy forces are taken into account. The laminar boundary layer and heat transfer along horizontally and vertically moving cylinders with constant velocity were examined by Lin and Shih (1980, 1981) who discovered that the cylinder's curvature effect prevented the similarity solutions from being reached. Because the primary differential equations governing fluid motion in hydrodynamics contain nonlinear components, an exact solution is necessary. It becomes challenging, if not impossible, to find the closed-form solution to such types of differential equations. This leads to the researchers arriving to obtain the solution for similarity. Researchers such as Chen and Char (1988), Wang (1981), Magyari and Keller (2000), Vajravelu and Cannon (2006), Ahmad et al. (2010), Bachok et al. (2012), Khan et al. (2012) and Begum et al. (2020) investigated these types of nonlinear problems using numerous numerical approaches such as Begum et al. (2023), Alam et al. (2020), Alam et al. (2021) and Alam et al. (2022) to find the solution.

In this work, we determine the velocity component of boundary layer flow past a stretching cylinder moving continuously at velocity $W(x) = \frac{W_0(x)}{l}$. Because of the nonlinearity present in the flow problem, we employ the numerical method known as finite difference method. Since the curvature parameter α = 0.0, 0.25, 0.5, 0.75, 1 affects the lateral surface of the cylinder, we have investigated the impact of velocity.

2. PROBLEM FORMULATION

Consider an axisymmetric, continuous boundary layer flow of a viscous incompressible fluid past a continuously stretched cylinder. The stretching velocity $W(x)$ is expressed as the relation $W(x) = \frac{W_0(x)}{l}$, where l is the characteristic length and $W_0 \succ 0$ is a constant.

With these presumptions along with the boundary layer estimations, the model equation can be written as follows

$$\frac{\partial}{\partial x}(rw) + \frac{\partial}{\partial r}(rv) = 0 \quad (1)$$

$$w\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \quad (2)$$

with boundary conditions (BCs)

$r = R, \quad w(0) = w(x), \; v(0) = 0, \quad (3)$

$r \to \infty, \quad w(r) \to 0\,, \quad (4)$

where the velocity components in the *x* and *r* directions are represented by the variables *w* and *v*, respectively. By implementing a stream function, χ, such that $u = \frac{1}{r}\frac{\partial \chi}{\partial r}$ and $v = -\frac{1}{r}\frac{\partial \chi}{\partial r}$, the continuity equation (1) can be satisfied. We define λ and χ as

$$\lambda = \frac{r^2 - R^2}{2R}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}},$$

$$\chi = R(U\nu x)^{1/2}\, V(\lambda),$$

so that the momentum equation becomes

$(1 + 2\alpha\lambda)V'''(\lambda) + 2\alpha V''(\lambda) + V(\lambda)V''(\lambda) - V'^2(\lambda) = 0, \quad (5)$

with relevant BCs:

$\lambda = 0, \quad V(\lambda) = 0, \; V'(\lambda) = 1, \quad (6)$

$\lambda \to \infty, \quad V'(\lambda) = 0 \quad (7)$

A non-linear boundary-value problem (bvp) in an infinite domain is illustrated by Equations (5) with (6) and (7). We solve this nonlinear bvp numerically using the finite difference method for various curvature parameters α, since there are no conventional methods for handling such problems.

3. SKIN FRICTION

To compute the surface sheer stress, let

$$\tau_w = -\mu\left(\frac{\partial u}{\partial r}\right)_{r=R}, \quad (8)$$

$$\text{or,} \quad \tau_w = -\mu \mathrm{U}\left(\frac{\mathrm{m}}{\gamma}\right)^{\frac{1}{2}} V''(0), \quad (9)$$

Hence, for the given (bvp), the skin friction co-efficient is

$$C_V = -\left(R_e{}^{-1}\right)\left(\frac{\mathrm{m}}{\gamma}\right)^{\frac{1}{2}} V''(0). \quad (10)$$

4. TRIGONOMETRIC QUARTIC SPLINE METHOD

To obtain trigonometric quartic spline approximation of the equations (5)-(7), we divide the interval $[a, b]$ into M equal subintervals as follows

$$\lambda_i = ih, \quad i = 0(1)M, \quad h = \frac{b-a}{M}$$

Now, using the non-polynomial spline $A_i(\lambda)$ we construct a numerical algorithm to interpolate the unknown function $V(\lambda)$ at the grid points $\{\lambda_i | i = 1,2,3 \ldots, M\}$ given as

$$A_i(\lambda) = C_{1i}\sin\omega(\lambda - \lambda_i) + C_{2i}\cos\omega(\lambda - \lambda_i) + C_{3i}(\lambda - \lambda_i)^2 + C_{4i}(\lambda - \lambda_i) + C_{5i}, \quad (11)$$

where C_{ji}, $j = 1,2,3,4,5$, are real finite constants and $A_i(\lambda)$ has been interpolated at the mesh points λ_i which depends on the parameter ω.

The coefficients C_{ji}, $j = 1,2,3,4,5$, have been obtained by using the following interpolation conditions:

$A_i(\lambda_i) = V_i, \; A'_i(\lambda_i) = E_i, \; A'''_i(\lambda_i) = F_i, \quad i = 0,1,2, \ldots, M \quad (12)$

Using the conditions given in equation (12) in the equation (11), we obtain the values of the co-efficients C_{ji}, $j = 1,2,3,4,5$. Further, following the continuity condition defined for spline as well as its derivatives, the relations have been obtained as:

$$E_i + E_{i+1} = -\frac{2(V_{i-1} - V_i)}{h} + \alpha_1 F_{i-1} + \alpha_1 F_i, \quad (13)$$

$$E_i - E_{i+1} = \frac{(V_{i-1} - 2V_i + V_{i+1})}{h} + \alpha_2 F_{i-1} + \alpha_3 F_i + \alpha_4 F_{i+1} \quad (14)$$

where

$$\alpha_1 = \frac{2 - 2\cos\omega h - h\omega\sin\omega h}{h\omega^3\sin\omega h},$$

$$\alpha_2 = \frac{2\cos\omega h + 2h\omega\sin\omega h - 2 - h^2\omega^2}{2h\omega^3\sin\omega h},$$

$$\alpha_3 = \frac{2h^2\omega^2\cos\omega h - 2h\omega\sin\omega h}{2h\omega^3\sin\omega h},$$

$$\alpha_4 = \frac{2 - 2\cos\omega h - h^2\omega^2}{2h\omega^3\sin\omega h}.$$

Solving the equations (13) and (14), we obtain the relation

$$-V_{i-2} + 3V_{i-1} - 3V_i + V_{i+1} = h^3(\xi_1 F_{i-2} + \xi_2 F_{i-1} + \xi_2 F_i + \xi_1 F_{i+1}), \quad i = 2(1)M-1 \quad (15)$$

where

$$\xi_1 = \frac{2 - 2cos\omega h - h^2\omega^2}{2\omega^3 sin\omega h},$$
$$\xi_2 = \frac{2h^2\omega^2 cos\omega h + 2cos\omega h - h^2\omega^2 - 2}{2h\omega^3 sin\omega h}.$$

The equations in (15) yield (M − 2) linear equations involving M unknowns in V_i, $i = 1,2,3, \dots, M$.
In order to solve the system of equations, we need two additional equations, which can be obtained as:

$$\sum_{k=0}^{2} \beta_{1k} V_k + \beta_2 h V_0' + h^3 \sum_{k=0}^{3} \beta_{3k} V_k''' - t_1 = 0, \quad i = 1 \qquad (16)$$

$$\sum_{k=M-2}^{M} \beta_{4k} V_k + \beta_5 h V_M' + h^3 \sum_{k=M-3}^{M} \beta_{6k} V_k''' - t_M = 0, \quad i = M. \qquad (17)$$

Now, implementing the above method in the equations (5)-(7), and with the help of Newton-Raphson method we find the approximate solution to (5)-(7), which is computed with the help of MATLAB.

5. NUMERICAL EXPERIMENTS AND DISCUSSIONS

Here, we study the results obtained by the proposed numerical method for the model prob- lem (5)-(7) at different grid points on the interval [0, 8]. MATLAB is used to produce a graphical depiction of the various components for different values of α. Figure 1 displays the numeri- cal findings of $V(\lambda)$ for various values of the parameter α. Additionally, Figure 3 provides a graphical depiction of $V'(\lambda)$ that illustrates the impact of the velocity component $V'(\lambda)$ when α fluctuates.

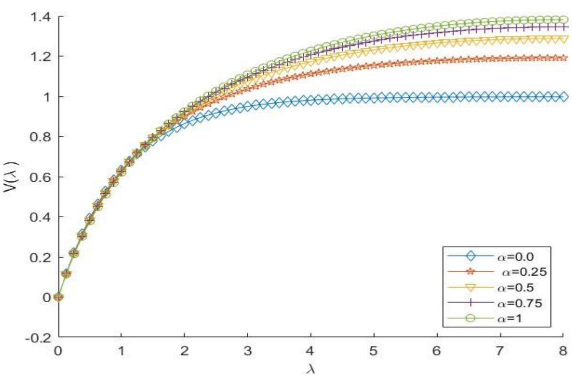

Figure 1: $V'(\lambda)$ *as the value of* α *varies* .

With reference to Figure 1, we observe that the horizontal velocity profile has not been affected by the curvature parameter inside the dynamic area [0, 1.5], following this, the velocity profile decreases as the curvature of the stretching cylinder reduces. The outside surface of the cylinder acts as a flat surface when we take $\alpha \to 0$. This indicates that as $\alpha \to 0$, the viscosity effect decreases because fluid-contact area of the surface moves toward the tangential position.

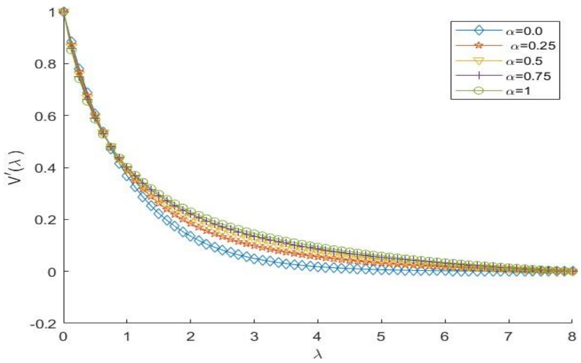

Figure 2: $V'(\lambda)$ *as the value of* α *varies* .

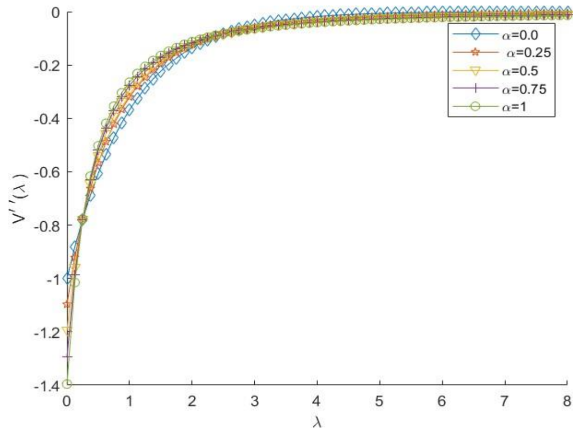

Figure 3: $V''(\lambda)$ *as the value of* α *varies* .

As we see in Figure 2, throughout the dynamic area [0, 1], the curvature parameter has essentially negligible influence on the horizontal velocity profile of the velocity field. Within the region $[1, \infty[$ the velocity component asymptotically approaches to zero. The velocity within $[1, \infty[$, in this case, is the free stream velocity and in this region as α increases, the velocity profile increases.

Figure 3 demonstrate the stress profile $V''(\lambda)$ as the parameters α varies. The coefficient $V''(0)$ for various values of α is displayed in Table 1. According to this table, the skin friction coefficient obtained by FDM agrees with the values obtained from Rangi et al. (2012). It is observed that the results agree well with earlier findings and hence the authenticity of our method to this problem is justified.

Table 1: Skin friction coefficient $V''(0)$ is compared using FDM with that of [26].

α	Rangi et al. (2012)	*Our Method*
0.00	0.927680	0.927683
0.25	1.232587	1.232587
0.50	1.477233	1.477231
0.75	1.232587	1.232587

6. CONCLUSION

In this chapter, we use finite difference approach to solve the boundary layer flow past a stretching cylinder. We employ finite difference method (FDM) is used to solve the problem for different values of parameter α. Furthermore, comparative study of the values of $V''(0)$ by FDM with that of values reported in Rangi et al. (2012) has been done in Table 1. Our approach produced a solution that is consistent with the one found in Rangi et al. (2012). Based on our approach, the results summarize that the curvature of the stretching cylinder is a crucial parameter that affects the flow.

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