

A Note on Continued Fractions

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Abstract: In this article we take a look at finite continued fractions and prove some theorems on Continued fractions.

Key words: Finite continued fraction, simple continued fraction, convergents, rational.

A very important application of the Euclidean algorithm lies in the continued fractions, which also gives an alternative way of representing real numbers. Let us begin with the numbers $a = 214$ and $b = 35$. By applying the Euclidean algorithm to these numbers we find

$$214 = 35 \cdot 6 + 4, \quad (1)$$

$$35 = 4 \cdot 8 + 3, \quad (2)$$

$$4 = 3 \cdot 1 + 1, \quad (3)$$

$$3 = 1 \cdot 3 + 0. \quad (4)$$

We now divide both sides of Equation (1) by 35, obtaining

$$\frac{214}{35} = 6 + \frac{4}{35} \quad (5)$$

So we have obtained a first piece of information: the rational number $214 / 35$ lies between 6 and 7, as $0 < 4 / 35 < 1$. By writing $4 / 35$ as the inverse of a number greater than 1, formula (5) becomes

$$\frac{214}{35} = 6 + \frac{1}{\frac{35}{4}} \quad (6)$$

$$\frac{35}{4} = 8 + \frac{3}{4} \quad \text{that is} \quad \frac{35}{4} = 8 + \frac{1}{\frac{4}{3}} \quad (7)$$

$$\frac{4}{3} = 1 + \frac{1}{3} \quad (8)$$

$$\frac{214}{35} = 6 + \frac{1}{8 + \frac{1}{1 + \frac{1}{3}}} \quad (9)$$

and the last expression is called a finite continued fraction.

Definition 1: Let a_0, a_1, \dots, a_n be real numbers, all positive except possibly a_0 . The expression

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

is called a finite continued fraction and is denoted by $[a_0; a_1, \dots, a_n]$. The numbers a_k are called the terms or the partial quotients of the continued fraction. The reason for assuming $a_k > 0$ for $k \geq 1$ in the above definition is that this guarantees that no division by zero will occur. A continued fraction is said to be simple if all of the a_i are integers.

Theorem 1: Every finite simple continued fraction is equal to a rational number, and every rational number can be written as a finite simple continued fraction.

Proof. The first part is trivial. For the second one, let a / b be the rational number, $b > 0$. Apply the Euclidean algorithm to find the gcd of a and b :

$$a = ba_0 + r_1, \quad 0 < r_1 < b,$$

$$b = r_1a_1 + r_2, \quad 0 < r_2 < r_1,$$

$$r_1 = r_2a_2 + r_3, \quad 0 < r_3 < r_2,$$

⋮

$$r_i = r_{i+1}a_{i+1} + r_{i+2}, \quad 0 < r_{i+2} < r_{i+1},$$

⋮

$$r_{n-2} = r_{n-1}a_{n-1} + r_n, \quad 0 < r_n < r_{n-1},$$

$$r_{n-1} = r_n a_n + 0.$$

As all the remainders are positive, so are all the quotients a_i , with the possible exception of the first one. Rewrite the equations given by the Euclidean algorithm dividing the first one by b , the second one by r_1 , the third one by r_2 and so on, till the last one, to be divided by r_n . So we obtain

$$\frac{a}{b} = a_0 + \frac{r_1}{b} = a_0 + \frac{1}{\frac{b}{r_1}},$$

$$\frac{b}{r_1} = a_1 + \frac{r_2}{r_1} = a_1 + \frac{1}{\frac{r_1}{r_2}},$$

$$\frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2} = a_2 + \frac{1}{\frac{r_2}{r_3}},$$

⋮

$$\frac{r_{n-1}}{r_n} = a_n.$$

The left-hand sides of these equations are rational numbers, which are rewritten as the sum of an integer and a fraction with numerator equal to 1. By successive eliminations, we get

$$\frac{a}{b} = a_0 + \frac{1}{\frac{1}{b}} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{1}{r_1}}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\frac{1}{r_2}}}};$$

until we obtain the expression

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}}.$$

So we have represented the rational number a/b as a finite simple continued fraction.

Let $[a_0; a_2, a_3, \dots, a_n]$ be a finite simple continued fraction. The continued fraction obtained by truncating this continued fraction after the k -th partial quotient is called k -th convergent and is denoted as follows:

$$C_k = [a_0; a_2, a_3, \dots, a_k], \text{ for each } 1 \leq k \leq n.$$

Notice that C_{k+1} may be obtained from C_k by substituting $a_k + \frac{1}{a_{k+1}}$ for a_k . Clearly, for $k = n$ we get the complete original continued fraction. Every $C_k = [a_0; a_1, \dots, a_k]$ is a rational number which will be denoted by p_k/q_k , where $\gcd(p_k, q_k) = 1$.

Suppose now that we have computed the value of $[a_0; a_1, a_2, \dots, a_n]$ and want to compute the value of $[a_0; a_1, a_2, \dots, a_{n+1}]$ without having to repeat the whole computation from scratch. The following recursion formula describes how to find $(n + 1)$ th convergent knowing n th convergent.

Theorem 2: If $a_0, a_1, a_2, \dots, a_n$ be real numbers with a_1, a_2, \dots positive. Let the sequences $p_0, p_1, p_2, \dots, p_n$ and $q_0, q_1, q_2, \dots, q_n$ be defined recursively by

$$\begin{aligned} p_{-1} &= q_{-2} = 1, \text{ and } p_{-2} = q_{-1} = 0, \\ p_0 &= a_0, \quad q_0 = 1, \\ p_1 &= a_0 a_1 + 1, \quad q_1 = a_1 \quad \& \\ p_k &= a_k p_{k-1} + p_{k-2} \quad \text{and } q_k \\ &= a_k q_{k-1} + q_{k-2} \quad \text{for } k \\ &= 2, 3, 4, \dots, n. \end{aligned}$$

Then the k th convergent is given by

$$C_k = \frac{p_k}{q_k}.$$

Proof: We will prove this by Mathematical Induction. For $k = 0$, we have

$$C_0 = [a_0] = \frac{p_0}{q_0}$$

For $k = 1$

$$C_1 = [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}$$

Therefore the Theorem is valid for $k = 0$ and $k = 1$.

Now, assume that the theorem is valid for k with $2 \leq k \leq n$. This means

$$C_k = [a_0, a_1, \dots, a_k] = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}.$$

Now, consider

$$\begin{aligned} C_{k+1} &= [a_0, a_1, \dots, a_k, a_{k+1}] \\ &= \left[a_0, a_1, \dots, a_k + \frac{1}{a_{k+1}} \right] \\ &= \frac{\left[a_k + \frac{1}{a_{k+1}} \right] p_{k-1} + p_{k-2}}{\left[a_k + \frac{1}{a_{k+1}} \right] q_{k-1} + q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{(a_{k+1} p_k + p_{k-1})}{(a_{k+1} q_k + q_{k-1})} = \frac{p_{k+1}}{q_{k+1}}. \end{aligned}$$

Example: We have $173 / 55 = [3; 6, 1, 7]$. Let us compute the sequences p_j and q_j for $j = 0, 1, 2, 3$. We have

$$\begin{aligned} p_0 &= 3, & q_0 &= 1 \\ p_1 &= 3 \cdot 6 + 1 = 19 & q_1 &= 6 \\ p_2 &= 1 \cdot 19 + 3 = 22 & q_2 &= 1 \cdot 6 + 1 = 7 \\ p_3 &= 7 \cdot 22 + 19 = 173 & q_3 &= 7 \cdot 7 + 6 = 55 \end{aligned}$$

$$C_0 = \frac{p_0}{q_0} = 3, \quad C_1 = \frac{p_1}{q_1} = \frac{19}{6}, \quad C_2 = \frac{p_2}{q_2} = \frac{22}{7}, \quad C_3 = \frac{p_3}{q_3} = \frac{173}{55}.$$

Theorem 3: If $a_0, a_1, a_2, \dots, a_n$ be real numbers with a_1, a_2, \dots positive, with corresponding convergent $C_n = \frac{p_n}{q_n}$. Then

- (i) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, if $n \geq -1$;
- (ii) $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$, if $n \geq 0$;
- (iii) $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$, if $n \geq 1$;
- (iv) $C_n - C_{n-2} = \frac{(-1)^n a_n}{q_{n-2} q_n}$, if $n \geq 2$.

Proof (i): Write $z_n = p_n q_{n-1} - p_{n-1} q_n$. Then $z_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1} (a_n q_{n-1} + q_{n-2}) = p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = -z_{n-1}$, for $n \geq 0$, and it follows at once that $z_n = (-1)^{n-1} z_{-1}$.

But $z_{-1} = 1$

Since $p_{-1} = q_{-2} = 1$, and $p_{-2} = q_{-1} = 0$. Hence $z_n = (-1)^{n-1}$ as required.

Proof (ii): Using the recursive definition of p_n and q_n and equality (i), we obtain

$$\begin{aligned} p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-2} \\ &\quad - p_{n-2} (a_n q_{n-1} + q_{n-2}) \\ &= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = \\ a_n (-1)^{n-2} &= (-1)^n a_n. \end{aligned}$$

(iii) follows from (i) upon division by $q_{n-1} q_n$, which is nonzero for $n \geq 1$.

(iv) follows from (ii) upon division by $q_{n-2} q_n$.

Theorem 4: Let a_0, a_1, a_2, \dots be real numbers with a_1, a_2, \dots positive, with corresponding convergents $C_n = \frac{p_n}{q_n}$. Then the convergents C_{2i} with even indices form a strictly increasing sequence and the convergents C_{2j+1} with odd indices form a strictly decreasing sequence, and $C_{2i} < C_{2j+1}$, that is

$$C_0 < C_2 < \dots < C_{2i} < \dots < C_{2j+1} < \dots < C_3 < C_1.$$

Proof: We have, $C_n - C_{n-2} = \frac{(-1)^n a_n}{q_{n-2} q_n}$. Hence if $n \geq 2$

is even, then $C_n - C_{n-2} > 0$ and if $n \geq 3$ is odd, then $C_n - C_{n-2} < 0$. Finally, by Theorem (iii), $C_{2k+1} - C_{2k} = \frac{1}{q_{2k} q_{2k+1}} > 0$. Thus if $i \geq j$, then $C_{2j} < C_{2i} < C_{2i+1}$ and $C_{2i} < C_{2i+1} < C_{2j+1}$.

In the above example,

$$3 < (22 / 7) < (173 / 55) < (19 / 6)$$

in accordance with $C_0 < C_2 < C_3 < C_1$.

Theorem 5: If q_k is the denominator of the k^{th} convergent C_k of the simple continued fraction $[a_0; a_1, a_2, \dots, a_n]$, then $q_k - 1 \leq q_k$ for $1 \leq k \leq n$, with strict inequality when $k > 1$.

Proof: We prove the theorem by induction. Since $q_0 = 1 \leq a_1 = q_1$, the theorem is true for $k = 1$. Assume that it is true for $k = m$ where $1 \leq m < n$. Then

$$q_{m+1} = a_{m+1} q_m + q_{m-1} > a_{m+1} q_m \geq q_m$$

So that the inequality is also true for $k = m+1$.

REFERENCES

- [1] Elementary Number Theory, David M. Burton, McGraw Hill Publication
- [2] An Introduction to the Theory of Numbers, G. H. Hardy and E. M. Wright, Oxford
- [3] Encyclopedia of Mathematics and its Applications, Volume 11, Continued Fractions, Analytic Theory and Applications, William B. Jones and W. J. Thron, Addison-Wesley
- [4] An Introduction to the Theory of Numbers, Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery, John Wiley & Sons, Inc.