A Note on Continued Fractions

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***Abstract:* In this article we take a look at finite continued fractions and prove some theorems on Continued fractions.**

***Key words:* Finite continued fraction, simple continued fraction, convergents, rational.**

A very important application of the Euclidean algorithm lies in the continued fractions, which also gives an alternative way of representing real numbers. Let us begin with the numbers $a = 214$ and $b = 35$. By applying the Euclidean algorithm to these numbers we find

$$214 = 35 \cdot 6 + 4, \qquad (1)$$

$$35 = 4 \cdot 8 + 3, \qquad (2)$$

$$4 = 3 \cdot 1 + 1, \qquad (3)$$

$$3 = 1 \cdot 3 + 0. \qquad (4)$$

We now divide both sides of Equation (1) by 35, obtaining

$$\frac{214}{35} = 6 + \frac{4}{35} \qquad (5)$$

So we have obtained a first piece of information: the rational number 214 / 35 lies between 6 and 7, as $0 < 4 / 35 < 1$. By writing 4 / 35 as the inverse of a number greater than 1, formula (5) becomes

$$\frac{214}{35} = 6 + \frac{1}{\frac{35}{4}} \qquad (6)$$

$$\frac{35}{4} = 8 + \frac{3}{4} \text{ that is } \frac{35}{4} = 8 + \frac{1}{\frac{4}{3}} \qquad (7)$$

$$\frac{4}{3} = 1 + \frac{1}{3} \qquad (8)$$

$$\frac{214}{35} = 6 + \frac{1}{8 + \frac{1}{1 + \frac{1}{3}}} \qquad (9)$$

and the last expression is called a finite continued fraction.

Definition 1: Let $a_0, a_1, \ldots, a_n$ be real numbers, all positive except possibly a_0. The expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

is called a finite continued fraction and is denoted by $[a_0; a_1, \ldots, a_n]$. The numbers a_k are called the terms or the partial quotients of the continued fraction. The reason for assuming $a_k > 0$ for $k \geq 1$ in the above definition is that this guarantees that no division by zero will occur. A continued fraction is said to be simple if all of the a_i are integers.

Theorem 1: Every finite simple continued fraction is equal to a rational number, and every rational number can be written as a finite simple continued fraction.

Proof. The first part is trivial. For the second one, let a / b be the rational number, $b > 0$. Apply the Euclidean algorithm to find the gcd of a and b:

$a = ba_0 + r_1,\ 0 < r_1 < b,$

$b = r_1 a_1 + r_2,\ 0 < r_2 < r_1,$

$r_1 = r_2 a_2 + r_3,\ 0 < r_3 < r_2,$

$\vdots$

$r_i = r_{i+1} a_{i+1} + r_{i+2},\ 0 < r_{i+2} < r_{i+1},$

$\vdots$

$r_{n-2} = r_{n-1} a_{n-1} + r_n,\ 0 < r_n < r_{n-1},$

$r_{n-1} = r_n a_n + 0.$

As all the remainders are positive, so are all the quotients a_i, with the possible exception of the first one. Rewrite the equations given by the Euclidean algorithm dividing the first one by b, the second one by r_1, the third one by r_2 and so on, till the last one, to be divided by r_n. So we obtain

$$\frac{a}{b} = a_0 + \frac{r_1}{b} = a_0 + \frac{1}{\frac{b}{r_1}},$$

$$\frac{b}{r_1} = a_1 + \frac{r_2}{r_1} = a_1 + \frac{1}{\frac{r_1}{r_2}},$$

$$\frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2} = a_2 + \frac{1}{\frac{r_2}{r_3}},$$

$$\vdots$$

$$\frac{r_{n-1}}{r_n} = a_n .$$

The left-hand sides of these equations are rational numbers, which are rewritten as the sum of an integer and a fraction with numerator equal to 1. By successive eliminations, we get

$$\frac{a}{b} = a_0 + \frac{1}{\frac{b}{r_1}} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{r_1}{r_2}}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\frac{r_2}{r_3}}}};$$

until we obtain the expression

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

So we have represented the rational number a / b as a finite simple continued fraction.

Let [a_0; a_2, a_3, . . . , a_n] be a finite simple continued fraction. The continued fraction obtained by truncating this continued fraction after the k-th partial quotient is called k-th convergent and is denoted as follows:

C_k= [a_0; a_2, a_3, . . . , a_k], for each $1 \le k \le n$.

Notice that C_{k+1} may be obtained from C_k by substituting $a_k + \frac{1}{a_{k+1}}$ for a_k. Clearly, for $k = n$ we get the complete original continued fraction. Every C_k = [a_0; a_1, . . . , a_k] is a rational number which will be denoted by p_k / q_k, where gcd(p_k, q_k) = 1.

Suppose now that we have computed the value of [a_0; a_1, a_2,…,a_n] and want to compute the value of [a_0; a_1, a_2,…,a_{n+1}] without having to repeat the whole computation from scratch. The following recursion formula describes how to find $(n + 1)^{th}$ convergent knowing n^{th} convergent.

Theorem 2: If a_0, a_1, a_2, . . . , a_n be real numbers with a_1, a_2, . . . positive. Let the sequences p_0, p_1, p_2, . . . , p_n and q_0, q_1, q_2, . . . , q_n be defined recursively by

$$p_{-1} = q_{-2} = 1 \text{, and } p_{-2} = q_{-1} = 0,$$
$$p_0 = a_0 , \quad q_0 = 1,$$
$$p_1 = a_0 a_1 + 1 , \; q_1 = a_1 \;\&$$
$$p_k = a_k p_{k-1} + p_{k-2} \text{ and } q_k = a_k q_{k-1} + q_{k-2} \quad \text{for } k = 2, 3, 4, \dots, n.$$

Then the k^{th} convergent is given by

$$C_k = \frac{p_k}{q_k}.$$

Proof: We will prove this by Mathematical Induction.
For $k = 0$, we have

$$C_0 = [a_0] = \frac{p_0}{q_0}$$

For $k = 1$

$$C_1 = [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}$$

Therefore the Theorem is valid for $k = 0$ and $k = 1$.
Now, assume that the theorem is valid for k with $2 \le k \le n$. This means

$$C_k = [a_0, a_1, \dots, a_k] = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}.$$

Now, consider

$$\begin{aligned} C_{k+1} &= [a_0, a_1, \dots, a_k, a_{k+1}] \\ &= \left[a_0, a_1, \dots, a_k + \frac{1}{a_{k+1}}\right] \\ &= \frac{\left[a_k + \frac{1}{a_{k+1}}\right] p_{k-1} + p_{k-2}}{\left[a_k + \frac{1}{a_{k+1}}\right] q_{k-1} + q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{(a_{k+1} p_k + p_{k-1})}{(a_{k+1} q_k + q_{k-1})} = \frac{p_{k+1}}{q_{k+1}}. \end{aligned}$$

Example: We have 173 / 55 = [3;6, 1, 7]. Let us compute the sequences p_j and q_j for j = 0, 1, 2, 3. We have

p_0 = 3,	q_0 = 1
p_1 = 3.6+1 = 19	q_1 = 6
p_2 = 1.19+3 = 22	q_2 = 1.6+1 = 7
p_3 = 7.22+19 = 173	q_3 = 7.7+6 = 55

$$C_0 = \frac{p_0}{q_0} = 3, \quad C_1 = \frac{p_1}{q_1} = \frac{19}{6}, \quad C_2 = \frac{p_2}{q_2} = \frac{22}{7}, \quad C_3 = \frac{p_3}{q_3} = \frac{173}{55}.$$

Theorem 3: If a_0, a_1, a_2, . . . , a_n be real numbers with a_1, a_2, . . . positive, with corresponding convergent $C_n = \frac{p_n}{q_n}$. Then

(i) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, if $n \ge -1$;

(ii) $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$, if $n \ge 0$;

(iii) $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$, if $n \ge 1$;

(iv) $C_n - C_{n-2} = \frac{(-1)^n a_n}{q_{n-2} q_n}$, if $n \ge 2$.

Proof (i): Write $z_n = p_n q_{n-1} - p_{n-1} q_n$. Then

$$z_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2})$$
$$= p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = -z_{n-1},$$ for $n \ge 0$, and it follows at once that $z_n = (-1)^{n-1} z_{-1}$.

But $z_{-1} = 1$

Since $p_{-1} = q_{-2} = 1$, and $p_{-2} = q_{-1} = 0$. Hence $z_n = (-1)^{n-1}$ as required.

Proof (ii): Using the recursive definition of p_n and q_n and equality (i), we obtain

$$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2}) = a_n(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) = a_n(-1)^{n-2} = (-1)^n a_n.$$

(iii) follows from (i) upon division by $q_{n-1}q_n$, which is nonzero for $n \geq 1$.

(iv) follows from (ii) upon division by $q_{n-2}q_n$.

Theorem 4: Let $a_0, a_1, a_2, \ldots$ be real numbers with $a_1, a_2, \ldots$ positive, with corresponding convergents $C_n = \frac{p_n}{q_n}$. Then the convergents C_{2i} with even indices form a strictly increasing sequence and the convergents C_{2j+1} with odd indices form a strictly decreasing sequence, and $C_{2i} < C_{2j+1}$, that is

$C_0 < C_2 < \ldots < C_{2i} < \ldots < C_{2j+i} < \ldots < C_3 < C_1$.

Proof: We have, $C_n - C_{n-2} = \frac{(-1)^n a_n}{q_{n-2}q_n}$. Hence if $n \geq 2$ is even, then $C_n - C_{n-2} > 0$ and if $n \geq 3$ is odd, then $C_n - C_{n-2} < 0$. Finally, by Theorem (iii), $C_{2k+1} - C_{2k} = \frac{1}{q_{2k}q_{2k+1}} > 0$. Thus if $i \geq j$, then $C_{2j} < C_{2i} < C_{2i+1}$ and $C_{2i} < C_{2i+1} < C_{2j+1}$.

In the above example,

$$3 < (22 / 7) < (173 / 55) < (19 / 6)$$

in accordance with $C_0 < C_2 < C_3 < C_1$.

Theorem 5: If q_k is the denominator of the k^{th} convergent C_k of the simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$, then $q_k - 1 \leq q_k$ for $1 \leq k \leq n$, with strict inequality when $k > 1$.

Proof: We prove the theorem by induction. Since $q_0 = 1 \leq a_1 = q_1$, the theorem is true for $k = 1$. Assume that it is true for $k = m$ where $1 \leq m < n$. Then

$$q_{m+1} = a_{m+1}q_m + q_{m-1} > a_{m+1}q_m \geq q_m$$

So that the inequality is also true for $k = m+1$.

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