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Some *P-Q* eta function identities

Dr. Madhusudhan H. S.
Assistant Professor Government First Grade College, Bannur, Mysore

Abstract: ***In unorganized portion of his second note book [3], Ramanujan recorded twenty-three results on P-Q eta function identities or modular equations. These are the identities involving the quotients of eta-function which are designated by P or Q by Ramanujan. Berndt [2] established proof of 18 of these identities by employing theory of theta functions in the spirit of Ramanujan and remaining 5 by employing the theory of modular forms. These modular equations play important role in the explicit evaluations of continued fractions and Class Invariants. In this paper we deduce certain P-Q modular equations and using these modular equations we obtain the values of Class Invariants.***

Keywords: Modular equations, theta functions, class-invariants, quotients of eta-functions

1. INTRODUCTION

Let, as usual,

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1,$$

and

$$\chi(q) := (-q;q^2)_\infty.$$

Ramanujan first introduced the Class Invariants

$$G_n = 2^{\frac{-1}{4}} e^{-\frac{\pi\sqrt{n}}{24}} \chi\left(e^{-\pi\sqrt{n}}\right)$$

and

$$g_n = 2^{\frac{-1}{4}} e^{-\frac{\pi\sqrt{n}}{24}} \chi\left(-e^{-\pi\sqrt{n}}\right),$$

in his famous paper, "Modular equations and approximation to π" [5]. In his first notebook [4], Ramunajan recorded the values for 107 Class Invariants or the polynomials satisfied by them. On pages 294 to 299 in his second notebook [3], Ramanujan gave a table of values of 77 Class Invariants, three of which are not found in his first notebook.

Motivated by these, in this paper we establish further evaluations of the class invariant g_n. In the next section, we establish some eta-function identities employing which we give in next section some general theorems for evaluating g_n. Using these general formulas, we derive many numerical Class Invariants.

Ramanujan's general theta function *f*(*a*, *b*) is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Furthermore, define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

We require the following definition of modular equation.

Definition: A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{F\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x) = \sum_{n=-\infty}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}x^k, \quad |x|<1$$

with

$$(a)_k = \frac{\gamma(a+k)}{\gamma(a)}.$$

β is said to have degree *n* over α.

2. *P-Q* ETA FUNCTION IDENTITIES

Theorem 2.1. If

$$P = q^{-\frac{1}{3}}\chi^2(-q)\chi^2(-q^3) \text{ and } Q = q^{-\frac{2}{3}}\chi^2(-q^2)\chi^2(-q^6)$$

then

$$Q^2 - P^2Q - 4P = 0. \tag{2.1}$$

Proof: We have from [1, p.223]

$$q\psi(q^2)\psi(q^6) = \frac{q}{1-q^2} - \frac{q^5}{1-q^{10}} + \frac{q^7}{1-q^{14}} - \frac{q^4}{1-q^{22}} + \cdots$$

and $\varphi(q)\varphi(q^3) = 1 + 2\left(\frac{q}{1-q} - \frac{q^2}{1+q^2} + \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5} + \cdots\right).$

From the above two identities, Berndt [1] established that

$$q\psi(q^2)\psi(q^6) = \varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3).$$

(2.2)

It is easy to verify that

$$\psi(q) = \frac{f(-q^2)}{\chi(-q)}, \qquad \varphi(q) = f(-q)\chi(-q),$$

$$\varphi(q) = \frac{\chi^2(-q^2)}{\chi^2(-q)} f(-q^2) \text{ and } \chi(-q) = \frac{f(-q)}{f(-q^2)}.$$

(2.3)

Using (2.3) in (2.2), we deduce the required result.

Theorem 2.2. If

$$P = q^{-1}\chi(-q)\chi(-q^5) \text{ and } Q = q^{-2}\chi^4(-q^2)\chi^4(-q^{10})$$

(2.4)

then

$$Q^2 - (P^2 + 8P)Q - 16P = 0.$$

Proof : If $y = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}$ and $q = e^{-y}$

we find that [1, p.124, entry 12, (vi), (vii)];

$$\chi(-q) = 2^{\frac{1}{6}}(1-x)^{\frac{1}{12}}x^{-\frac{1}{24}}q^{\frac{1}{24}},$$

and

$$\chi(-q^2) = 2^{\frac{1}{3}}(1-x)^{\frac{1}{24}}x^{-\frac{1}{12}}q^{\frac{1}{24}}.$$

Using the above two identities, we deduce that

$$(1-x)^{\frac{1}{8}} = \frac{\chi^2(-q)}{\chi(-q^2)}$$

(2.5)

and

$$x^{\frac{1}{8}} = \sqrt{2}q^{\frac{1}{8}}\frac{\chi^2(-q)}{\chi(-q^2)}.$$

(2.6)

Let β is of degree 5 over α. If

$$\alpha^{\frac{1}{2}} = 4q^{\frac{1}{2}}\frac{\chi^4(-q)}{\chi^8(-q^2)}$$

then from (2.5) and (2.6), it follows that

$$\beta^{\frac{1}{2}} = 4q^{\frac{5}{2}}\frac{\chi^4(-q^5)}{\chi^8(-q^{10})}, \qquad (1-\alpha)^{\frac{1}{2}} = \frac{\chi^8(-q)}{\chi^4(-q^2)},$$

and

$$(1-\beta)^{\frac{1}{2}} = \frac{\chi^8(-q^5)}{\chi^8(-q^{10})}.$$

From [1, p.280, Entry 13(i)], we see that

$$(\alpha\beta)^{\frac{1}{2}} + [(1-\alpha)(1-\beta)]^{\frac{1}{2}} + 2[16\alpha\beta(1-\alpha)(1-\beta)]^{\frac{1}{6}} = 1.$$

Substituting above values, we deduce that

$$16q^3\frac{\chi^4(-q)\chi^4(-q^5)}{\chi^8(-q^2)\chi^4(-q^{10})} + \frac{\chi^8(-q)\chi^8(-q^5)}{\chi^4(-q^2)\chi^4(-q^{10})} + 8q\frac{\chi^4(-q)\chi^4(-q^5)}{\chi^4(-q^2)\chi^4(-q^{10})} = 1.$$

Thus

$$Q^2 - (P^2 + 8P)Q - 16P = 0.$$

Hence the proof.

Theorem 2.3. If

$$P = q^{-\frac{1}{3}}\chi(-q)\chi(-q^5) \text{ and } Q = q^{-\frac{2}{3}}\chi(-q^2)\chi(-q^{14}),$$

Then

$$Q^2 - P^2Q - 2P = 0.$$

(2.7)

Proof: We have from [1, p.315]

$$\varphi(q)\varphi(q^7) = 2q\psi(q)\psi(q^7) + \varphi(-q^2)\varphi(-q^{14}).$$

(2.8)

Using (2.3) in (2.8), we find that

$$\frac{2q}{\chi(-q)\chi(-q^7)} + \chi(-q^2)\chi(-q^{14}) = \frac{\chi^2(-q^2)\chi^2(-q^{14})}{\chi^2(-q)\chi^2(-q^7)}$$

and then after doing some algebraic manipulation, we obtain

$$Q^2 - P^2Q - 2P = 0.$$

3. CLASS INVARIANTS

In this section, we give many interesting general formulas for evaluating the product of class invariants. Employing general formulas so obtained we give further values of g_n.

Theorem 3.1 We have

$$g_{2n}\, g_{2/n} = 1$$

(3.1)

Proof: From entry 27 (iii) of chapter 16 of Ramanujan's second notebook [4], [1, p.124], we have, if $\alpha\beta = \pi^2$ then

$$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}).$$

(3.2)

Consider

$$g_{2n}\, g_{2/n} = 2^{-1/2}\, e^{\frac{-\pi}{24}\left[\sqrt{2n}+\sqrt{\frac{2}{n}}\right]}\chi\left(e^{-\pi\sqrt{2n}}\right)\chi\left(e^{-\pi\sqrt{2/n}}\right)$$

$$= 2^{-1/2}\, e^{\frac{-\pi}{24}\left[\sqrt{2n}+\sqrt{\frac{2}{n}}\right]}\frac{f\left(-e^{-\pi\sqrt{2n}}\right)}{f\left(-e^{-2\pi\sqrt{2n}}\right)}\cdot\frac{f\left(-e^{-2\pi\sqrt{2/n}}\right)}{f\left(-e^{-2\pi\sqrt{2/n}}\right)}$$

$$= 2^{-1/2}\, e^{\frac{-\pi}{24}\left[\sqrt{2n}+\sqrt{\frac{2}{n}}\right]}\frac{f\left(-e^{-\pi\sqrt{2n}}\right)}{f\left(-e^{-2\pi\sqrt{2n}}\right)}\cdot\frac{f\left(-e^{-\pi\sqrt{2/n}}\right)}{f\left(-e^{-2\pi\sqrt{2/n}}\right)}.$$

Using (3.2) in the above, we find that

$$g_{2n}\, g_{2/n} = 2^{-1/2}\, e^{\frac{-\pi}{24}\left[\sqrt{2n}+\sqrt{\frac{2}{n}}\right]} e^{\frac{1}{12}\left[\pi\sqrt{\frac{n}{2}}-\pi\sqrt{\frac{2}{n}}\right]} \left(\frac{2}{n}\right)^{1/4} \times e^{\frac{1}{12}\left[\frac{\pi}{\sqrt{2n}}-\pi\sqrt{2n}\right]}(2n)^{1/4}$$

Hence the proof.

Theorem 3.2. We have

$$g_{4n}^4\, g_{36n}^4 - 2g_n^4\, g_{9n}^4\, g_{4n}^2\, g_{36n}^2 - 2g_n^2\, g_{9n}^2 = 0$$

(3.3)

Proof: Setting $q = e^{-\pi\sqrt{n}}$ in Theorem 2.1 and then using the definition of g_n we find that
$P = 2g_n^2\, g_{9n}^2$ and $= 2g_{4n}^2\, g_{36n}^2$, where *P* and *Q* are as in Theorem 2.1. Employing these in (2.1), we obtain.

$$g_{4n}^4\, g_{36n}^4 - 2g_n^4\, g_{9n}^4\, g_{4n}^2\, g_{36n}^2 - 2g_n^2\, g_{9n}^2 = 0.$$

Theorem 3.3. We have

$$g_{4n}^8\, g_{100n}^8 - 4g_n^4\, g_{25n}^4\, g_{4n}^4\, g_{100n}^4(g_n^4\, g_{25n}^4 + 2) - 4g_n^4\, g_{25n}^4 = 0$$

(3.4)

Proof: Setting $q = e^{-\pi\sqrt{n}}$ in Theorem 2.2 and then using the definition of g_n we find that $P = 4g_n^4\, g_{25n}^4$ and $= 4g_{4n}^4\, g_{100n}^4$, where *P* and *Q* are as in Theorem 2.2. Employing these in (2.4), we obtain.

$$g_{4n}^8\, g_{100n}^8 - 4g_n^4\, g_{25n}^4\, g_{4n}^4\, g_{100n}^4(g_n^4\, g_{25n}^4 + 2) - 4g_n^4\, g_{25n}^4 = 0.$$

Theorem 3.4. We have

$$g_{4n}^2\, g_{196n}^2 - \sqrt{2}g_n\, g_{49n}(g_n\, g_{49n}\, g_{4n}\, g_{196n} + 1) = 0.$$

(3.5)

Proof: Setting $q = e^{-\pi\sqrt{n}}$ in Theorem 2.3 and then employing the definition of g_n we find that $P = \sqrt{2}g_n\, g_{49n}$ and $= \sqrt{2}g_{4n}\, g_{196n}$, where *P* and *Q* are as in Theorem 2.3. Employing these in (2.7), we obtain

$$G_{4n}^2\, g_{196n}^2 - \sqrt{2}g_n\, g_{49n}(g_n\, g_{49n}\, g_{4n}\, g_{196n} + 1) = 0.$$

Hence the result.

Corollary 3.1. We have

$$g_{3/2}g_{1/6} = \sqrt{\frac{-1+\sqrt{3}}{2}}$$

Proof: Setting $n = 1/6$ in (3.3), we deduce that

$$2g_{1/6}^4\, g_{3/2}^4\, g_{2/3}^2\, g_6^2 + 2g_{1/6}^2\, g_{3/2}^2 - g_{2/3}^4\, g_6^4 = 0.$$

Setting $n = 3$ in (3.1) we have $g_6 g_{2/3} = 1$. Using this in the above identity, we find that

$$2g_{1/6}^4\, g_{3/2}^4 + 2g_{1/6}^2\, g_{3/2}^2 - 1 = 0.$$

Solving the above equation for $g_{1/6}g_{3/2}$, we obtain

$$g_{1/6}g_{3/2} = \sqrt{\frac{-1\pm\sqrt{3}}{2}}$$

Since g_n is positive for all rational *n*,

$$g_{1/6}g_{3/2} = \sqrt{\frac{-1+\sqrt{3}}{2}}.$$

Corollary 3.2. We have

$$g_8\, g_{72} = \frac{\sqrt{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}}{\sqrt[6]{\sqrt{3}-\sqrt{2}}}$$

Proof: Putting $n = 2$ in (3.3), we obtain

$$g_8^4\, g_{72}^4 - 2g_2^4\, g_{18}^4\, g_8^2\, g_{72}^2 - 2g_2^2\, g_{18}^2 = 0$$

From [3, P.200], we have $g_2 = 1$ and $g_{18} = (\sqrt{2}+\sqrt{3})^{1/3}$. Employing these in the above identity, we find that

$$g_8^4\, g_{72}^4 - 2(\sqrt{2}+\sqrt{3})^{4/3} g_8^2\, g_{72}^2 - 2(\sqrt{2}+\sqrt{3})^{2/3} = 0$$

On solving the above equation for $g_8 g_{72}$, we deduce that

$$g_8^4\, g_{72}^4 = (\sqrt{2}+\sqrt{3})^{1/6}\left(\sqrt{2}+\sqrt{3} \pm \sqrt{7+2\sqrt{6}}\right)^{1/2}.$$

Using the fact that is positive for all rational *n* and after some algebraic manipulation we find that

$$g_8 g_{72} = \frac{\sqrt{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}}{\sqrt[6]{\sqrt{3}-\sqrt{2}}}$$

Corollary 3.3. We have

$$g_{1/2}g_{9/2} = \frac{\sqrt{-1+2\sqrt{2}+\sqrt{3}}}{\sqrt{2}\,\sqrt[3]{\sqrt{2}+\sqrt{3}}}.$$

Proof: Setting $n = 1/2$ in (3.3), we obtain

$$2g_{1/2}^4\, g_{9/2}^4\, g_2^2\, g_{18}^2 + 2g_{1/2}^2\, g_{9/2}^2 - g_2^4\, g_{18}^4 = 0$$

From [3, P.200], we have $g_2 = 1$ and $g_{18} = (\sqrt{2}+\sqrt{3})^{1/3}$. Using these in the above identity we find that

$$2(\sqrt{2}+\sqrt{3})^{2/3} g_{1/2}^4\, g_{9/2}^4 + 2g_{1/2}^4\, g_{9/2}^2 - (\sqrt{2}+\sqrt{3})^{4/3} = 0$$

On solving the above equation for $g_{1/2}g_{9/2}$ we find that

$$g_{1/2}g_{9/2} = \frac{\sqrt{-1 \pm \sqrt{11+4\sqrt{6}}}}{\sqrt{2}\sqrt[3]{\sqrt{2}+\sqrt{3}}}$$

Since g_n is positive for all rational n and after some algebraic manipulation we find that

$$g_{1/2}g_{9/2} = \frac{\sqrt{-1 + \sqrt{11+4\sqrt{6}}}}{\sqrt{2}\sqrt[3]{\sqrt{2}+\sqrt{3}}}$$

Corollary 3.4. We have

$$g_{5/2}g_{45/2} = \left(\frac{-1+\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{\frac{1}{2}}.$$

where

$$a = \sqrt{\frac{1+\sqrt{5}}{2}},$$

and

$$b = \left\{(2+\sqrt{5})(\sqrt{5}+6)\right\}^{1/6}\left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}}\right).$$

Proof: Putting $n = 5/2$ in (3.3), we obtain

$$2\, g_{10}^2\, g_{90}^2\, g_{5/2}^4\, g_{45/2}^4 + 2\, g_{5/2}^2\, g_{45/2}^2 - g_{10}^4\, g_{90}^4 = 0.$$

From [3, P.200, 202] we have $g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}$ and

$$g_{90} = \left\{(2+\sqrt{5})(\sqrt{5}+6)\right\}^{1/6}\left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}}\right).$$

Employing these in the above identity, we find that

$$2a^2b^2\, g_{5/2}^4\, g_{45/2}^4 + 2\, g_{5/2}^2\, g_{45/2}^2 - a^4b^4 = 0.$$

On solving the above equation for $g_{5/2}\, g_{45/2}$, we deduce that

$$g_{5/2}\, g_{45/2} = \left(\frac{-1\pm\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{1/2}.$$

Since g_n is positive for all rational n, we find that

$$g_{5/2}\, g_{45/2} = \left(\frac{-1+\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{1/2}$$

Corollary 3.5 we have

$$g_{5/2}\, g_{45/2} = \left(\frac{-1+\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{1/2}$$

where

$$a = \sqrt{\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}}$$

and

$$b = \sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(\sqrt{6}+\sqrt{7})^{1/6}\left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}\right)^2$$

Proof: setting $n = 7/2$ in (3.3), we deduce that

$$2\, g_{14}^2\, g_{126}^2\, g_{7/2}^4\, g_{63/2}^4 + 2\, g_{7/2}^2\, g_{63/2}^2 - g_{14}^4\, g_{126}^4 = 0$$

From [3, P.200, 202], we have

$$g_{14} = \sqrt{\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}}$$

and

$$g_{126} = \sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(\sqrt{6}+\sqrt{7})^{1/6}\left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}\right)^2$$

Using these in the above identity, we find that

$$2\, a^2b^2\, g_{7/2}^4\, g_{63/2}^4 + 2\, g_{7/2}^2\, g_{63/2}^2 - a^4b^4 = 0.$$

Solving the above equation for $g_{7/2}\, g_{63/2}$, we find that

$$g_{7/2}\, g_{63/2} = \left(\frac{-1\pm\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{1/2}.$$

Since g_n is positive for all rational n, we find that

$$g_{7/2}\, g_{63/2} = \left(\frac{-1+\sqrt{1+2a^6b^6}}{2a^2b^2}\right)^{1/2}.$$

Corollary 3.6 we have

$$g_{1/10}\, g_{5/2} = \left(\frac{-3+\sqrt{10}}{2}\right)^{1/4}.$$

Proof: Setting $n = \frac{1}{10}$ in (3.4), we obtain

$$2g_{2/5}^{8}\, g_{10}^{8} - 4\, g_{1/10}^{8}\, g_{5/2}^{8}\, g_{2/5}^{4}\, g_{10}^{4} - 8g_{1/10}^{4}\, g_{5/2}^{4}\, g_{2/5}^{4}\, g_{10}^{4} - 4g_{1/10}^{4}\, g_{5/2}^{4} = 0.$$

Setting $n = 5$ in (3.1) we have $g_{10}\, g_{2/5} = 1$. Using this in the above identity, we find that

$$4\, g_{1/10}^{8}\, g_{5/2}^{8} + 12 g_{1/10}^{4}\, g_{5/2}^{4} - 1 = 0.$$

Solving the above equation for $g_{1/10} g_{5/2}$, we obtain

$$g_{1/10} g_{5/2} = \left(\frac{-3 \pm \sqrt{10}}{2}\right)^{1/4}$$

Since g_n is positive for all rational *n*, we find that

$$g_{1/10} g_{5/2} = \left(\frac{-3+\sqrt{10}}{2}\right)^{1/4}.$$

Corollary 3.7 we have

$$g_{1/2}\, g_{25/2} = \frac{1}{a}\left(\frac{-1 - 2a^4 + \sqrt{1 + 4a^4 + 4a^8 + +a^{12}}}{2}\right)^{1/4},$$

where

$$a = \frac{1}{3}\left(1 + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1 + 7\sqrt{5} + 6\sqrt{6}} + \sqrt[3]{1 + 7\sqrt{5} - 6\sqrt{6}}\right)\right).$$

Proof: Setting $n = 1/2$ in (3.4), we deduce that

$$4g_{1/2}^{8}\, g_{25/2}^{8} g_{2}^{4}\, g_{50}^{4} + 8\, g_{1/2}^{4}\, g_{25/2}^{4}\, g_{2}^{4}\, g_{50}^{4} + 4g_{1/2}^{4}\, g_{25/2}^{4} - g_{2}^{8}\, g_{50}^{8} = 0.$$

From [3, P.200, 201], we have $g_2 = 1$ and

$$g_{50} = \frac{1}{3}\left(1 + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1 + 7\sqrt{5} + 6\sqrt{6}} + \sqrt[3]{1 + 7\sqrt{5} - 6\sqrt{6}}\right)\right).$$

Employing these in above identity, we find that

$$4a^4 g_{1/2}^{8}\, g_{25/2}^{8} + 4(2a^4 + 1) g_{1/2}^{4}\, g_{25/2}^{4} - a^8 = 0.$$

Solving the above equation for $g_{1/2}\, g_{25/2}$, we find that

$$g_{1/2}\, g_{25/2} = \frac{1}{a}\left(\frac{-1 - 2a^4 \pm \sqrt{1 + 4a^4 + 4a^8 + +a^{12}}}{2}\right)^{1/4}$$

Since g_n is positive for all rational *n*, we find that

$$g_{1/2}\, g_{25/2} = \frac{1}{a}\left(\frac{-1-2a^4+\sqrt{1+4a^4+4a^8+a^{12}}}{2}\right)^{1/4}.$$

Corollary 3.8: We have

$$g_8\, g_{200} = \sqrt{a}\,\sqrt[4]{2a^2(a^4 + 2) + \sqrt{1 + 4a^4 + 4a^8 + a^{12}}},$$

where

$$a = \frac{1}{3}\left(1 + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1 + 7\sqrt{5} + 6\sqrt{6}} + \sqrt[3]{1 + 7\sqrt{5} - 6\sqrt{6}}\right)\right).$$

Proof: Setting $n = 2$ in (3.4), we deduce that

$$g_{8}^{8}\, g_{200}^{8} - 4g_{2}^{8}\, g_{50}^{8}\, g_{8}^{4}\, g_{200}^{4} - 8g_{2}^{4}\, g_{50}^{4}\, g_{8}^{4}\, g_{200}^{4} - 4g_{2}^{4}\, g_{50}^{4} = 0.$$

From [3, P.200, 201], we have $g_2 = 1$ and

$$g_{50} = \frac{1}{3}\left(1 + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1 + 7\sqrt{5} + 6\sqrt{6}} + \sqrt[3]{1 + 7\sqrt{5} - 6\sqrt{6}}\right)\right)$$

Using these in above identity, we find that

$$g_{8}^{8}\, g_{200}^{8} - 4(a^4 + 2) g_{8}^{4}\, g_{200}^{4} - 4a^4 = 0.$$

Solving the above equation for $g_8\, g_{200}$, we find that

$$g_8\, g_{200} = \sqrt{a}\,\sqrt[4]{2a^2(a^4 + 2) \pm \sqrt{a^{12} + 4a^8 + 4a^4 + 1}}$$

Since g_n is positive for all rational *n*, we find that

$$g_8\, g_{200} = \sqrt{a}\,\sqrt[4]{2a^2(a^4 + 2) + \sqrt{a^{12} + 4a^8 + 4a^4 + 1}}.$$

Corollary 3.9 we have

$$g_{1/14}\, g_{7/2} = \left(\frac{-2+\sqrt{2+4\sqrt{2}}}{2\sqrt{2}}\right).$$

Proof: Setting $n = 1/14$ in (3.5), we deduce that

$$\sqrt{2}g_{1/14}^2\, g_{7/2}^2 g_{2/7} g_{14} + \sqrt{2}g_{1/14} g_{7/2} - g_{2/7}^2\, g_{14}^2 = 0.$$

On setting $n = 7$ in (3.1) we have $g_{14} g_{2/7} = 1$.
Using this in the above identity, we find that

$$\sqrt{2}g_{1/14}^2\, g_{7/2}^2 + \sqrt{2}g_{1/14}^2\, g_{7/2}^2 - 1 = 0.$$

Solving the above relation for $g_{14} g_{7/2}$, we obtain

$$g_{1/14}\, g_{7/2} = \left(\frac{-2\pm\sqrt{2+4\sqrt{2}}}{2\sqrt{2}}\right)$$

Since g_n is positive for all rational n, we find that

$$g_{1/14}\, g_{7/2} = \left(\frac{-2+\sqrt{2+4\sqrt{2}}}{2\sqrt{2}}\right).$$

Corollary 3.10 we have

$$g_8\, g_{392} = \frac{a^2+\sqrt{a^4+2\sqrt{2}a}}{\sqrt{2}},$$

where

$$a = \left(\sqrt{\frac{4+\sqrt{2}+\sqrt{14+4\sqrt{14}}}{8}} + \sqrt{\frac{\sqrt{2}+\sqrt{14+4\sqrt{14}-4}}{8}}\right)^4.$$

Proof: Putting $n = 2$ in (3.5), we deduce that

$$g_8^2\, g_{392}^2 - \sqrt{2}g_2^2\, g_{98}^2 g_8\, g_{392} - \sqrt{2}g_2\, g_{98} = 0.$$

From [3, P.200, 202], we have $g_2 = 1$ and

$$g_{98} = \left(\sqrt{\frac{4+\sqrt{2}+\sqrt{14+4\sqrt{14}}}{8}} + \sqrt{\frac{\sqrt{2}+\sqrt{14+4\sqrt{14}-4}}{8}}\right)^2.$$

Employing these in the above identity, we find that

$$g_8^2\, g_{392}^2 - \sqrt{2}a^2 g_8\, g_{392} - \sqrt{2}a = 0.$$

Solving the above relation for $g_8\, g_{392}$, and after doing some algebraic manipulation we obtain

$$g_8\, g_{392} = \frac{a^2\pm\sqrt{a^4+2\sqrt{2}a}}{\sqrt{2}}.$$

Since g_n is positive for all rational n,

$$g_8\, g_{392} = \frac{a^2+\sqrt{a^4+2\sqrt{2}a}}{\sqrt{2}}.$$

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