Findings on Complex Valued b-Metric Spaces Subject To Contractive Condition

[1]Raksha Dwivedi, [2]Dr. Richa Gupta
[1] Research Scholar (Department of Mathematic) Sarvepalli Radhakrishnan University, Bhopal (M.P.), India Pin code 462026
[2](Supervision) Professor, Department of Mathematics Sarvepalli Radhakrishnan University, Bhopal, India Pin code 462026

Abstract: This Paper establishes a common fixed-point theorem for two self-mappings in a complex Valued b-metric space under specific contractive conditions. The result extends the findings of S. Ali [3].

Keywords: Complex Valued b-metric Space, Common Fixed-Point, Contractive Type Mapping.

1.INTRODUCTION

The concept of Complex Valued Metric Spaces was introduced by A. Fisher and M. Khan [4] in 2011. The idea of b-Metric Spaces dated back to Bakhtin [5] in 1989. Rao et al. [9] later combined these ideas to develop complex-valued b-metric spaces, which generalize the standard complex-valued metric spaces. Numerous fixed-point theorems have been established in the context of complex-valued metric spaces [2], [6], [7], [8], [10], [11], as well as in complex- valued b-metric spaces [1], [9]. In this paper we provide a common fixed-point theorem for two self-mappings satisfying a contractive condition in complex-valued b-metric spaces. This Result builds on and generalizes the work of S. Ali [3].

2.PRELIMINARIES

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_1)$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_1)$
(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_1)$
(3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_1)$
(4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_1)$

We write $z_1 \precsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 < z_2$ if only (4) is Satisfied.

Remark: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz, \forall z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \precsim z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Azam et al. [4] defined the complex valued metric space in the following way:

Definition 2.1 ([4]):

Let X be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.1 [7]: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = i|z_1 - z_2| \qquad \forall z_1, z_2 \in \mathbb{C}$$

One can easily verify that $(\mathbb{R}, d)$ is a complex valued metric space.

Definition 2.2 ([9]):
Let X be a nonempty set and let $\omega \geq 1$ be given real number. A function $d: X \times X \rightarrow \mathbb{C}$ is called

a complex valued b-metric on X, if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$
(3) $d(x,y) \precsim \omega[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X, d) is called complex valued b-metric space.

Example 2.2 [9]: Let $X = [0,1]$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$d(x,y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$.

Then (X, d) is a complex valued b-metric space with $k = 2$

Definition 2.3 ([9]): Let (X, d) be a complex valued b-metric space. Then

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ Such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ if for every $0 \prec r \in \mathbb{C}$ Such that $B(x,r) \cap (A - \{x\}) \neq \emptyset$.
A subset $A \subseteq X$ is called closed if each element of $X - A$ is not a limit point of A.

(iii) The family $F = \{B(x,r) : x \in X, 0 \prec r\}$ Is a sub-basis for a Hausdorff topology τ on X.

Definition 2.4 ([9]): Let (X, d) be a complex valued b-metric space. Then

⇒A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $x < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < r$, $\forall n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

⇒ If for every $0 < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < r$ for all $n > N, m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).

⇒ If every Cauchy sequence in X is convergent in X then (X, d) is called a complete complex valued b-metric space.

Lemma 2.1([9]): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.2([9]): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$ where $m \in \mathbb{N}$.

Definition 2.5 ([11]): The 'max' function for the partial order $\precsim$ is defined as follows:

(i) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.
(ii) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.
(iii) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

3.MAIN RESULT

In this section we present the main result of the paper.

Theorem 3.1 Let (X, d) be a complete complex valued b- metric space with coefficient

$\omega \geq 1$ and $S, T: X \to X$ be self-maps satisfying the following condition:

$$d(Sx, Ty) \leq \alpha \max\left\{d(x,y), \frac{d(y,Ty)[1 + d(x,Sx)]}{1 + d(x,y)}\right\} + \beta \max\left\{d(Sx,Ty), \frac{d(x,Sx)[1 + d(y,Sx)]}{1 + d(y,Ty).d(y,Sx)}\right\}$$

$\forall x, y \in X$, Where α and β are real with $0 < \alpha, 0 < \beta$, $\alpha + \beta < 1$ then S and T have a unique common fixed-point.

Proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ in X as

$x_{2k+1} = Sx_{2k}$

$x_{2k+2} = Tx_{2k+1}$, $k = 0,1,2, \dots\dots\dots$

Then $d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$

$$\leq \alpha \max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k+1}, Tx_{2k+1})[1 + d(x_{2k}, Sx_{2k})]}{1 + d(x_{2k}, x_{2k+1})}\right\} + \beta \max\left\{d(Sx_{2k}, Tx_{2k+1}),, \frac{d(x_{2k}, Sx_{2k})[1 + d(x_{2k+1}, Sx_{2k})]}{1 + d(x_{2k+1}, Tx_{2k+1}). d(x_{2k+1}, Sx_{2k})}\right\}$$

$$\leq \alpha \max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k+1}, x_{2k+2})[1 + d(x_{2k}, x_{2k+1})]}{1 + d(x_{2k}, x_{2k+1})}\right\} + \beta \max\left\{d(x_{2k+1}, x_{2k+2}),, \frac{d(x_{2k}, x_{2k+1})[1 + d(x_{2k+1}, x_{2k+1})]}{1 + d(x_{2k+1}, x_{2k+2}). d(x_{2k+1}, x_{2k+1})}\right\}$$

$$\leq \alpha \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} + \beta \max\{d(x_{2k+1}, x_{2k+2}), d(x_{2k}, x_{2k+1})\}$$

$d(x_{2k+1}, x_{2k+2}) \leq (\alpha + \beta) \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}$

$d(x_{2k+1}, x_{2k+2}) \leq (\alpha + \beta) M_1$

Where $M_1 = \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}$

Case-I: If $M_1 = d(x_{2k}, x_{2k+1})$

$d(x_{2k+1}, x_{2k+2}) \leq (\alpha + \beta) d(x_{2k}, x_{2k+1})$ (1)

Case-II: If $M_1 = d(x_{2k+1}, x_{2k+2})$

$d(x_{2k+1}, x_{2k+2}) \leq (\alpha + \beta) d(x_{2k+1}, x_{2k+2})$

$d(x_{2k+1}, x_{2k+2})(1 - \alpha - \beta) \leq 0$

$\because (1 - \alpha - \beta) \neq 0$

So $d(x_{2k+1}, x_{2k+2}) = 0$

$x_{2k+1} = x_{2k+2}$

We will continue with case-I

$d(x_{2k+2}, x_{2k+3}) \leq (\alpha + \beta) d(x_{2k+1}, x_{2k+2})$ (2)

Therefore from eq. (1) and (2) for $n \in N$ we have

$d(x_{n+1}, x_{n+2}) \leq (\alpha + \beta) d(x_n, x_{n+1}) \leq (\alpha + \beta)^2 d(x_{n-1}, x_n) \leq \cdots \leq (\alpha + \beta)^{n+1} d(x_0, x_1).$

So, for $m, n \in N$,

$d(x_n, x_{m+n}) \leq \omega[d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+n})]$

$\leq \omega[d(x_n, x_{n+1}) + \omega^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{m+n})]$

$\leq \omega d(x_n, x_{n+1}) + \omega^2 d(x_{n+1}, x_{n+2}) + \cdots + \omega^{m-1} d(x_{m+n-2}, x_{m+n-1}) + \omega^m d(x_{m+n-1}, x_{m+n})$

$d(x_n, x_{m+n}) \leq \omega(\alpha+\beta)^n d(x_0, x_1) + \omega^2(\alpha+\beta)^{n+1} d(x_0, x_1) + \cdots + \omega^m(\alpha+\beta)^{m+n-1} d(x_0, x_1)$

$d(x_n, x_{m+n}) \leq \omega(\alpha+\beta)^n [1 + \omega(\alpha+\beta) + \{\omega(\alpha+\beta)\}^2 + \cdots + \{\omega(\alpha+\beta)\}^{m-1}]\, d(x_0, x_1)$

$\to 0$ as $n \to \infty$ Where $m \in \mathbb{N}$.

Therefore, from Lemma 2, we see that $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete $\exists\ u \in X$ Such that $x_n \to u$ as $n \to \infty$

Thus $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u$

Now from the given condition we have

$$d(Su, u) \leq \omega[\, d(Su, Tx_{2n+1}) + d\,(Tx_{2n+1}, u)]$$

$$\leq \omega[\alpha \max\left\{d(u, x_{2n+1}), \frac{d(x_{2n+1}, Tx_{2n+1})[1 + d(u, Su)]}{1 + d(u, x_{2n+1})}\right\} + \beta \max\left\{d(Su, Tx_{2n+1}),, \frac{d(u, Su)[1 + d(x_{2n+1}, Su)]}{1 + d(x_{2n+1}, Tx_{2n+1}).\, d(x_{2n+1}, Su)}\right\}] + \omega\, d(Tx_{2n+1}, u)$$

$\to 0$ as $n \to \infty$

Thus $d(Su, u) \leq 0$

$\Rightarrow d(Su, u) = 0$

Hence $Su = u$

Similarly

$\Rightarrow d(Tu, u) = 0$

Hence $Tu = u$

Therefore, u is a common fixed-point of S and T.

Now for the uniqueness

Let us suppose that $Su^* = Tu^* = u^*$ for some $u^* \in X$.

Then $d(u, u^*) = d(Su, Tu^*)$

$$\leq \alpha \max\left\{d(u, u^*), \frac{d(u^*, Tu^*)[1 + d(u, Su)]}{1 + d(u, u^*)}\right\} + \beta \max\left\{d(Su, Tu^*), \frac{d(u, Su)[1 + d(u^*, Su)]}{1 + d(u^*, Tu^*).\, d(u^*, Su)}\right\}]$$

$$d(u, u^*) \leq \alpha \max\left\{d(u, u^*), \frac{d(u^*, u^*)[1 + d(u, u)]}{1 + d(u, u^*)}\right\} + \beta \max\left\{d(u, u^*), \frac{d(u, u)[1 + d(u^*, u)]}{1 + d(u^*, u^*).\, d(u^*, u)}\right\}]$$

$d(u, u^*) \leq (\alpha+\beta) d(u, u^*)$

$d(u, u^*)\,(1 - \alpha - \beta) \leq 0$

$\because (1 - \alpha - \beta) \neq 0$

$d(u, u^*) = 0$ Hence $u = u^*$

This completes the proof.

Corollary: - Let (X, d) be a complete complex valued b- metric space with coefficient $\omega \geq 1$ and $F, S: X \to X$ be self-maps satisfying the following condition:

$$d(Fx, Sy) \leq \alpha \max\left\{d(x, y), \frac{\{d(Fx, Sy)\}^2}{d(Sy, Fx).\, d(Fx, x) + d(Fx, Sy)}\right\}$$

$\forall\, x, y \in X$, Where α are real with $0 < \alpha < 1$ then S and F have a unique common fixed-point.

Declaration of Competing Interest

The authors affirm that they have no financial interests, personal relationships or other potential conflicts that might have influenced the findings presented in this study.

Data Availability

No data was utilized in the research described in the article.

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