W-Energy of Induced Semigraphs of Paths and Cycles

Ravikumar N.
Department of Mathematics, Government First Grade College, Gundulpet - 571111, India

Abstract-The W-energy of a semigraph G is the sum of the absolute values of its W-eigenvalues. In this paper we show that the graph energy of the classes of paths and cycles is more than the W-energy of the semigraphs when some vertices are replaced by middle vertices.

AMS Subject classification: 05C50, 05C99

Keywords: Weighted adjacency matrix; W-Energy; W-Characteristic polynomial.

1. INTRODUCTION

The concept of semigraph is a natural generalization of the graph introduced by E. Sampathkumar [7] wherein an edge may contain more than two vertices having middle vertices apart from the usual end vertices. A semigraph G is a pair (V, X), where V is a non-empty set whose elements are called vertices of G and X is a set of ordered n-tuples called edges of G of distinct vertices, for various $n \geq 2$, satisfying the following conditions:

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_n)$ are equal if and only if

(i) $m = n$ and

(ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Thus the edge $(u_1, u_2, \ldots, u_n)$ is the same as $(u_n, u_{n-1}, \ldots, u_1)$.

If $E = (v_1, v_2, \ldots, v_n)$ is an edge of a semigraph, we say that v_1 and v_n are the *end* vertices of edge E and v_i for $2 \leq i \leq n-1$ are the middle vertices or m-vertices of the edge E and also the vertices $v_1, v_2, \ldots, v_n$ are said to belong to the edge E. Two vertices u and $v, u \neq v$, in a semigraph are adjacent if both of them belong to the same edge.

An edge containing at least one m-vertex is called an S-edge, otherwise it is called an ordinary edge. A semigraph with p vertices and q edges is called a (p, q) semigraph. A partial edge of an edge $E = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ is a $(k-j+1)$ tuple $E' = (v_{i_j}, v_{i_{j+1}}, \ldots, v_{i_k})$ where $1 \leq j < k \leq n$. We say that, the partial edge E' has cardinality $k-j+1$, which we again denoted by $|E'|$. A subedge of an edge $E = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ is a k-tuple $E' = (v_{i_{j_1}}, v_{i_{j_2}}, \ldots, v_{i_{j_k}})$ where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

The number of vertices belonging to an edge E is called the cardinality of E and is denoted by $|E|$. A partial edge of cardinality 2 is called a unit partial edge. The length of an edge E is the number of unit partial edges of the edge E and is denoted by $l(E)$. Thus if $E = (v_1, v_2, \ldots, v_k)$ then $l(E) = k-1$ and $|E| = k$. The length of a partial edge is defined similarly. If the vertices u, v are adjacent, then $(u, \ldots, v)$ is a partial edge whose length is denoted by $l(u, v)$.

Three vertices v_i, v_j and v_k are said to form a triangle in a semigraph G, if they are pairwise adjacent but do not lie on the same edge. If the vertices v_i, v_j and v_k form a triangle in a semigraph then the partial edges $(v_i, \ldots, v_j), (v_j, \ldots, v_k)$ and $(v_k, \ldots, v_i)$ are called the sides of the triangle.

Let G be a simple graph with k vertices and $A(G)$ be its adjacency matrix. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $A(G)$. Then the *energy* of G [2], denoted by $E(G)$, is defined as $E(G) = \sum_{i=1}^{k} |\lambda_i|$. The energy of the cycle C_k and the path P_k are given by

$$E(C_k) = \begin{cases} \dfrac{4\cos\frac{\pi}{k}}{\sin\frac{\pi}{k}}, & \text{if } k \equiv 0 (\text{mod} 4); \\ \dfrac{4}{\sin\frac{\pi}{k}}, & \text{if } k \equiv 2 (\text{mod} 4); \\ \dfrac{2}{\sin\frac{\pi}{2k}}, & \text{if } k \equiv 1 (\text{mod} 2). \end{cases}$$

$$E(P_k) = \begin{cases} \dfrac{2}{\sin \dfrac{\pi}{2(k+1)}} - 2, & \text{if } k \equiv 0(\text{mod}2); \\ \dfrac{2\cos \dfrac{\pi}{2(k+1)}}{\sin \dfrac{\pi}{2(k+1)}} - 2, & \text{if } k \equiv 0(\text{mod}2). \end{cases}$$

A semigraph may have edges having several vertices including possible middle vertices apart from two end vertices. The L-adjacency matrix $L(G) = (l_{ij})$ of a semigraph G was defined in [6] to reflect this aspect by defining

$$l_{ij} = \begin{cases} 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent or } v_i = v_j; \\ l(v_i, v_j) & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \end{cases}$$

where two vertices v_i and v_j are adjacent if they belong to the same edge.

Given a (p, q)-semigraph G, we define its weighted adjacency (or W-adjacency) matrix $W(G) = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent or } v_i = v_j; \\ \dfrac{l(v_i, v_j)}{k} & \text{if } v_i \text{ and } v_j \text{ are adjacent lying on an edge of length } k. \end{cases}$$

Note that $W(G)$ is a symmetric $p \times p$ matrix with entries from $\mathbb{Q}$, the field of rationals. Also, if v_i, v_j are end vertices of an edge E, then $w_{ij} = \frac{l(v_i, v_j)}{l(E)} = 1$. Let $\mu_1, \dots, \mu_k$ be the eigenvalues of $W(G)$ (called W-eigenvalues). The W-energy $E_W(G)$ of the semigraph G is the sum of the absolute values of its W-eigenvalues i.e., $E_W(G) = \sum_{i=1}^{k} |\mu_i|$.

In section 2, we show that the graph energy of the classes of paths and cycles is more than the W-energy of the semigraphs when some vertices are replaced by middle vertices.

2. RESULTS

By introducing m number of middle vertices to each edge of the cycle C_n, the induced semigraph obtained is denoted by $C_{n,m}$. Similarly, by introducing m number of middle vertices to each edge of the path P_n, the induced semigraph obtained is denoted by $P_{n,m}$. We compare the energy of C_n and P_n with the W-energy of the induced semigraphs by making use of the upper bound for W energy of semigraphs obtained in Theorem. 3.7 of [5]: For a (p, q)-semigraph G, we have, $E_W(G) \leq \sqrt{-2ph_2}$,

where $-h_2 = \sum_{E \in X} \frac{(|E|)^2(|E|+1)}{12(|E|-1)}$.

Theorem 2.1: Suppose $k \geq 9$ and $k \equiv 3(\text{mod}6)$. Let $C_{k/3,2}$ be the semigraph with k vertices having $\frac{k}{3}$ number of edges each of cardinality 4 as shown below.

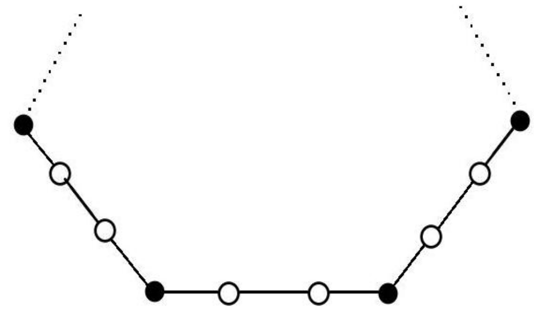

Fig. 2.1 Semigraph $C_{k/3,2}$

Then $E_W(C_{k/3,2}) < E(C_k)$.

Proof: By Theorem. 3.7 of [5], it is enough to show that $E(C_k) > \sqrt{-2h_2k}$ as $\sqrt{-2h_2k}$ is an upper bound for $E_W(C_{k/3,2})$, where $-h_2$ is the coefficient of η^{k-2} in the W-characteristic polynomial of $C_{k/3,2}$ is given by $-h_2 = \sum_{E \in X} \frac{|E|^2(|E|+1)}{12(|E|-1)}$. Now, the semigraph $C_{k/3,2}$ has $\frac{k}{3}$ number of edges each edge of cardinality 4 and so, we have,

$$-h_2 = \frac{k}{3} \times \frac{16 \times 5}{12 \times 3} = \frac{20}{27}k$$

and so

$$\sqrt{-2h_2k} = \sqrt{2 \times \frac{20k}{27} \times k} = k\sqrt{\frac{40}{27}}$$

Since k is odd, $E(C_k) = \frac{2}{\sin \frac{\pi}{2k}}$ and so it is enough to show $\frac{2}{\sin \frac{\pi}{2k}} > k\sqrt{\frac{40}{27}}$ for all $k \geq 9$ and $k \equiv 3(\text{mod}6)$.

We prove that $\frac{1}{\sin \frac{\pi}{2x}} > \frac{x}{3}\sqrt{\frac{10}{3}}$ for all $x \geq 9$.

Put $t = \frac{\pi}{2x}$. Then $\frac{x}{3}\sqrt{\frac{10}{3}} = \frac{\pi}{6t}\sqrt{\frac{10}{3}} = \frac{\alpha}{t}$, where $\alpha = \frac{\pi}{6}\sqrt{\frac{10}{3}} \approx 0.942$. Now we need to show $\frac{1}{\sin\, t} > \frac{\alpha}{t}$ for all $t \leq \frac{\pi}{18} = \gamma$, where $\gamma \approx 0.174$. Since $\sin\, t < t$ for all $t > 0$, we have, $t > \sin\, t > \alpha \sin\, t$ for all $t > 0$ as $0 < \alpha < 1$. Thus, we have, $E_W(C_{k/3,2}) < E(C_k)$, completing the proof of the theorem.

Theorem 2.2: Suppose $k \equiv 4(\mathrm{mod}6)$ and $P_{\frac{k+2}{3},2}$ be the semigraph with k vertices having $\frac{k-1}{3}$ number of edges each of cardinality 4 as given below:

Fig. 2.2 Semigraph $P_{\frac{k+2}{3},2}$

Then $E_W\left(P_{\frac{k+2}{3},2}\right) < E(P_k)$.

Proof: Using Theorem. 3.7 of [5], it is enough to show that $E(P_k) > \sqrt{-2h_2k}$ as $\sqrt{-2h_2k}$ is an upper bound for $E_W\left(P_{\frac{k+2}{3},2}\right)$, where $-h_2$ is the coefficient of η^{k-2} in the W-characteristic polynomial of $P_{\frac{k+2}{3},2}$ given by $-h_2 = \sum_{E\in X} \frac{|E|^2(|E|+1)}{12(|E|-1)}$. Since k is even, we have, $E(P_k) = \frac{2}{\sin\frac{\pi}{2(k+1)}} - 2$. Thus, it is enough to show that $\frac{2}{\sin\frac{\pi}{2(k+1)}} - 2 > \sqrt{-2h_2k}$. Now, the semigraph $P_{\frac{k+2}{3},2}$ has $\frac{k-1}{3}$ number of edges each of cardinality 4 . If we put $k = 6l + 4$, then $\frac{k-1}{3} = 2l + 1$ and so, we have,

$$-h_2 = (2l+1) \times \frac{16\times 5}{12\times 3} = (2l+1)\frac{20}{9}$$
$$= (6l+3)\frac{20}{27}$$

and so

$$\sqrt{-2h_2k} = \sqrt{(6l+3)\times\frac{40}{27}\times(6l+4)}$$
$$< (6l+4)\sqrt{\frac{40}{27}}$$

since $6l + 3 < 6l + 4$.

If we show that

$$E(P_k) = \frac{2}{\sin\frac{\pi}{2(6l+5)}} - 2 > (6l+4)\sqrt{\frac{40}{27}}$$

for all $l \geq 2$, then this would mean that $E(P_k) > \sqrt{-2h_2k}$ for all k with $k \geq 16$ and $k \equiv 4(\mathrm{mod}6)$.

We show that $\frac{1}{\sin\frac{\pi}{2(6x+5)}} > 1 + (6x+4)\sqrt{\frac{10}{27}}$ for all $x \geq 2$. Put $t = \frac{\pi}{2(6x+5)}$. Then, we have,

$$1 + (6x+4)\sqrt{\frac{10}{27}} = 1 + \left(\frac{\pi}{2t} - 1\right)\sqrt{\frac{10}{27}}$$
$$= \left(\sqrt{\frac{10}{27}} \times \frac{\pi}{2}\right)\frac{1}{t} + \left(1 - \sqrt{\frac{10}{27}}\right)$$
$$= \frac{\alpha}{t} + \beta, \text{ say,}$$

where $\alpha = \frac{\pi}{2}\sqrt{\frac{10}{27}} \approx 0.955$ and $\beta = 1 - \sqrt{\frac{10}{27}} \approx 0.3915$. Also as $x \geq 2$, we have, $t = \frac{\pi}{2(6x+5)} \leq \frac{\pi}{34} = \gamma$, say, where $\gamma \approx 0.09235$.

Now, $t > \sin\ t$ for all $t > 0$. Hence for $0 < t \leq \gamma$, we have,

$$\frac{1}{\sin\ t} > \frac{1}{t} = \frac{\alpha}{t} + \frac{1-\alpha}{t},$$

where

$$\frac{1-\alpha}{t} \geq \frac{1-\alpha}{\gamma} > \beta,$$

as $\frac{1-\alpha}{\gamma} \approx 0.484$. Thus,

$$\frac{1}{\sin\ t} > \frac{\alpha}{t} + \beta \text{ for } t \in (0, \gamma]$$

Thus $E(P_k) > E_W\left(P_{\frac{k+2}{3},2}\right)$ for all positive integers $k \geq 16$ such that $k \equiv 4(\mathrm{mod}6)$.

For $k = 4, 10$ (that is when $l = 0, 1$), it can easily be checked to show the validity of the theorem and so this proves the theorem.

REFERENCE

[1] D. M. Cvetkovi´c, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[2] I. Gutman, X. Li and Y. Shi, *Graph Energy*, New York: Springer, 2012.

[3] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, New York, 1969

[4] Y.Hou, *Unicyclic graphs with minimal energy*, Journal of Mathematical Chemistry, Vol. 29, No. 3, (2001), 163-168.

[5] H. N. Ramaswamy and N. Ravikumar, *Weighted Adjacency Matrix of a Semigraph and its Spectral Analysis*, Advanced Studies in Contemporary Mathematics, 27 (2017), No. 1, pp. 7 – 30.

[6] H.N. Ramaswamy and K.S. Shambhulingaiah, Adjacency matrices of semigraphs and their spectral analysis. Adv. Stud. Contemp. Math., 24 (2014), No. 3, pp. 349-368.

[7] E. Sampathkumar. Semigraphs and their applications, Technical Report [DST/MS/022/94] Department of Science and Technology, Govt of India, August; 1999.
[8] E. Sampathkumar, L. Pushpalatha, Matrix representation of Semigraphs, Adv. Stud. Contemp. Math., 14 (2007), No.1, pp. 103-109.