Computing Exact Solution for Linear Partial Differential Equation by Adomian Decomposition Method and Variational Iteration Method

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Abstract— In the present paper a new kind of Numerical technique is used for computing exact solution to a linear partial differential equation by method of Adomian decomposition and Variation iteration methods are described and used to give exact solution for some well-known nonlinear problem. In this method the problems are initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory. Finally, we compare two methods with their exact solution

Indexed Terms- Adomian decomposition method, Variational iteration method, Goursat problem, linear partial differential equation, Successive approximation

I. INTRODUCTION

In this paper we will study the Goursat problem [7] that arises in linear and non-linear partial differential equations with mixed derivatives. Several numerical methods [6] such as Range - Kutta method, finite difference method, finite elements method and geometric means averaging of the functional values of $f(x, y, u, u_x, u_y)$ have been used to approach the problem. However, the linear and non linear Goursat models will be approached more effectively and rapidly by using the Adomian decomposition method. The linear examples will be handled by the Variational iteration method as well. The Goursat problem in its standard form is given by

$$u_{xy} = f(x, y, u, u_x, u_y), 0 \le x \le a, 0 \le y \le b, u(x, 0) = g(x), u(0, y) = h(y), g(0) = h(0) = u(0,0). \quad (1.1)$$

$$L_x L_y u = f(x, y, u, u_x, u_y) \quad (1.2)$$

where $L_x = \frac{\delta}{\delta x}$, $\quad L_y = \frac{\delta}{\delta y}$

The inverse operators L_x^{-1} and L_y^{-1} can be defined as

$$L_x^{-1}(\cdot) = \int_0^x (\cdot)dx, L_y^{-1}(\cdot) = \int_0^y (\cdot)dy,$$

Because the Goursat problem (1.1) involves two distinct differential operators L_x and L_y two inverse integral operatorL_x^{-1} and L_y^{-1} will be used.

Applying L_y^{-1} to both sides of (1.2) gives

$$L_x[L_y^{-1} L_y u(x, y)] = L_y^{-1} f(x, y, u, u_x, u_y)$$

It then follows that

$$L_x[u(x, y) - u(x, 0)] = L_y^{-1} f(x, y, u, u_x, u_y),$$

or equivalently

$$L_x u(x, y) = L_x u(x, 0) + L_y^{-1} f(x, y, u, u_x, u_y) \quad (1.3)$$

Operating with L_x^{-1} on (1.3) yields

$$L_x^{-1} L_x u(x, y) = L_x^{-1} L_x u(x, 0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y),$$

This gives,

$$u(x, y) = u(x, 0) + u(0, y) - u(0,0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y),$$

or equivalently

$$u(x, y) = g(x) + h(y) - g(0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y) \quad (1.4)$$

Obtained upon using the conditions given in (1.1). Substituting $u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$ into (1.4) leads to

$$\sum_{n=0}^{\infty} u_n(x, y) = g(x) + h(y) - g(0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y)$$

Adomian's method admits the use of the recursive relation

$$u_0(x, y) = \eta(x, y) \text{ and } (x, y) = L_x^{-1} L_y^{-1} \sigma(u, u_x, u_y), k \ge 0 \quad (1.5)$$

where,

$$\eta(x, y) = \begin{cases} g(x) + h(y) - g(0), & f = \sigma(u, u_x, u_y) \\ g(x) + h(y) - g(0) + L_x^{-1} L_y^{-1} \tau(x, y), & f = \tau(x, y) + \sigma(u, u_x, u_y) \end{cases}$$

In view of (1.5) the solution in a series form follows immediately. The resulting series solution may provide the exact solution. Otherwise, the n-term approximation Φ_n can be used for numerical purposes. It can be shown that the difference between the exact solution and the n-terms approximation decreases monotonically for all values of x and y as additional components are evaluated. In the following one linear Goursat model will be discussed for illustrative purposes.

Consider the linear Goursat problem

$$u_{xy} = -x + u \qquad (1.6)$$

Subject to the conditions

$$u(x,0) = x + e^x, u(0,y) = e^y, u(0,0) = 0.$$

II. ADOMIAN DECOMPOSITION METHOD

For solving partial differential equation [5, 7], solutions are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems. To apply the Adomian decomposition method [1] for such non-linear partial differential equation [2, 7] The Decomposition method, using eq.(1.4), we find

$$u(x,y) = x + e^x + e^y - 1 - \frac{1}{2}x^2y + L_x^{-1}L_y^{-1}u(x,y) \qquad (2.1)$$

and by using the series representation for $u(x,y)$ into eq.(1.6) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = x + e^x + e^y - 1 - \frac{1}{2}x^2y + L_x^{-1}L_y^{-1}\sum_{n=0}^{\infty} u(x,y) \qquad (2.2)$$

The recursive relation

$$u_0(x,y) = x + e^x + e^y - 1 - \frac{1}{2}x^2y \qquad u_{k+1}(x,y) = L_x^{-1}L_y^{-1}u_k(x,y) \qquad k \geq 0 \qquad (2.3)$$

follows immediately consequently the first three components of the solution $u(x,y)$ are given by

$$u_0(x,y) = x + e^x + e^y - 1 - \frac{1}{2}x^2y$$
$$u_1(x,y) = L_x^{-1}L_y^{-1}u_0(x,y)$$
$$= \frac{1}{2}x^2y + y(e^x - 1) + x(e^y - 1) - xy - \frac{1}{12}x^3y^2$$
$$u_2(x,y) = L_x^{-1}L_y^{-1}u_1(x,y)$$
$$= \frac{1}{12}x^3y^2 + \frac{1}{2}y^2(e^x - 1 - x) + \frac{1}{2}x^2(e^y - 1 - y) - \frac{1}{4}x^2y^2 - \frac{1}{144}x^3y^4$$

This gives,

$$u(x,y) = x + e^x\left(1 + y + \frac{1}{2!}y^2\right) + \frac{1}{3!}y^3 + \cdots\Big) + e^y\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) - \left(1 + x + y + xy + \frac{1}{2!}x^2 + \frac{1}{2!}y^2 + \frac{1}{3!}x^3 + \frac{1}{3!}y^3 + \frac{1}{2!}x^2y + \cdots\right)$$

or equivalently

$$u(x,y) = x + e^x\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots\right) + e^y\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)\left(1 + y + \frac{1}{2!}y^2\right) + \frac{1}{3!}y^3 + \cdots\Big)$$

Accordingly, the solution in a closed form is given by

$$u(x,y) = x + e^{x+y}$$

Obtained upon using the Taylor's expansion for e^x and e^y.

III. VARIATIONAL ITERATION METHOD

The Variational iteration method (VIM)[3, 4, 5] gives rapidly convergent successive approximations of the exact solution if an exact solution exists. The obtained approximations by this method are of high accuracy level even if few iteration used. As introduced before the method employs the correction functional.

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\xi)\left(\frac{\delta^2 u_n(x,\xi)}{\delta x \delta \xi} - \tilde{u_n}(x,\xi) + x\right)d\xi \quad (3.1)$$

The stationary condition, $1 + \lambda = 0$, it follows that $\lambda' = 0$, which gives $\lambda = -1$. Substituting the Lagranges multiplier $\lambda = -1$, into the correction functional gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left(\frac{\delta^2 u_n(x,\xi)}{\delta x \delta \xi} - u_n(x,\xi) + x\right)d\xi, \quad n \geq 0 \qquad (3.2)$$

Selecting $u_0(x,y) = x + Ae^x + Be^y$ gives the following successive approximations

$$u_0(x,y) = x + Ae^x + Be^y$$
$$u_1(x,y) = x + Ae^x(1 + y) + 2Be^y - B$$
$$u_2(x,y) = x + Ae^x\left(1 + y + \frac{1}{2!}y^2\right) + 4Be^y - 3B - B_y$$

$u_3(x,y) = x + Ae^x\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3\right) +$
$8Be^y - 7B - 4B_y - \frac{1}{2}By^2$
$u_n(x,y) = x + Ae^x\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots\right) +$
$\left(8Be^y - 7B - 4B_y - \frac{1}{2}By^2 + \cdots\right)$
Using the boundary conditions $u(0,0) = 1$ and $u(x,o) = x + +e^x$ gives the system,

$$A + B = 1$$

$$x + Ae^x + B = x + e^x$$

Solving this system gives $A = 1, B = 0$. Substituting these values into $u_n(x,y)$ gives the exact solution
$u(x,y) = x + e^{x+y}$
obtained upon the using the Taylor's expansion for e^y.

The variational iteration method [3,4,5] will be used to handle non-linear problems in a manner similar to that used before for linear problems. The method facilitates the computational work for non-linear problems compared to Adomian method. Unlike Adomian decomposition method, the variational iteration method does not require specific treatment for non-linear operators. There is no need for Adomian polynomials that require a huge size of computational work. Moreover, the variational iteration method does not require specific assumption or restrictive conditions as required by other method such as perturbation techniques. The effectiveness and the efficiency of the method can be confirmed by discussing the following non-linear ordinary differential equations.

CONCLUSION

In this paper we have workout exact solution for Goursat problem of linear partial differential equation by using Adomian decomposition method and variational iteration method with some numerical techniques. This result shows that
(1) A correctional functional can be easily constructed by a general Lagrange multiplier, and the multiplier can be optimally identified by variational theory.
(2) The initial approximation can be freely selected with unknown constants, which can be determined via various methods.
(3) Comparison with Adomian decomposition method reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Adomian's.

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