A Study on Linear Algebra (Eigen Values and Eigen Vectors)

K. JYOTHI

Assistant Professor, Department of Mathematics, Nishitha Degree College (Autonomous), Affiliated to Telangana University, Nizamabad

***Abstract*— Fundamental ideas in linear algebra, eigenvalues and eigenvectors have broad applications in many fields of mathematics, science, and engineering. This abstract examines the basic ideas of eigenvalues and eigenvectors, highlighting the importance of these concepts in comprehending complicated systems and occurrences. Square matrices include two mathematical characteristics known as eigenvalues and eigenvectors. The scalar values known as eigenvalues show how an eigenvector is scaled by a matrix. On the other hand, eigenvectors are unique vectors that, while being scaled by the appropriate eigenvalue, continue to point in the same direction after being multiplied by the matrix. Comprehension diagonalization and linear transformations requires a comprehension of these essential ideas.There are many uses for eigenvalues and eigenvectors. They are used in physics to characterize quantum states, stability in dynamical systems, and oscillation modes. They are essential to structural analysis, control theory, and signal processing in engineering.**

***Index Terms*- Analyzing, Eigenvalues, Eigenvectors, Linear, Transformations**

I. INTRODUCTION

Linear algebra is one of the important branches of mathematics. Linear algebra is basically the study of vectors and linear functions. It is a key concept for almost all areas of mathematics. Linear algebra is considered a basic concept in the modern presentation of geometry. It is widely utilized in Physics and Engineering to represent fundamental concepts such as planes, lines, and the rotation of objects. This approach aids in modeling various natural phenomena while ensuring computational efficiency. In this article, we will explore an introduction to the topic, its key components, related problems, linear equations, and practical applications.

Applications in Engineering and Physics The physical sciences and engineering fields use eigenvalues and eigenvectors in many different ways. They are now essential tools in many domains due to their usefulness in modeling and comprehending complicated systems of mechanical systems.Eigenvalues and eigenvectors are utilized to the study of electrical circuits and systems in electrical engineering. They are used to investigate the behavior of linear time-invariant systems, such as filters and electrical networks. Engineers can determine the transient and steady-state responses of electrical systems to diverse inputs using eigenvalues and eigenvectors

Chemistry: Eigenvalues and Eigenvectors are employed in quantum chemical computations to solve the Schrödinger equation for molecular systems in computational chemistry. The electronic structure and energy levels of molecules may be predicted by chemists by determining the eigenvalues and eigenvectors of the molecular Hamiltonian. Understanding chemical processes and molecular characteristics requires the knowledge of these information.Fundamental ideas in linear algebra, eigenvalues and eigenvectors have many applications in mathematics, science, and engineering. They are useful tools for comprehending and interacting with complicated phenomena because of their capacity to disclose the fundamental properties of linear transformations, matrices, and systems. Either used in conjunction or separately, eigenvalues and eigenvectors provide a coherent framework for assessing and resolving a broad variety of real-world issues, whether they are related to computational chemistry, structural engineering, control systems, or quantum physics. The beauty and relevance of eigenvalues and eigenvectors in mathematics and beyond will be revealed as we dive deep into their characteristics, computations, and applications as we begin this thorough investigation of these important mathematical topics[9]

II. SIGNIFICANCE OF LINEAR ALGEBRA

- Linear algebra is the study of linear combinations. It involves the study of vector spaces, lines, planes, and specific mappings essential for performing linear transformations.
- It includes vectors, matrices and linear functions. It is the study of linear sets of equations and its transformation properties.

Linear Function:
A linear function is an algebraic equation where each term is either a constant or the product of a constant and a single independent variable raised to the power of 1. In linear algebra, vectors are used to construct linear functions. Various types of vectors can also be expressed in terms of vector functions.

Mathematically, linear function is defined as:
A function $L : R^n \rightarrow R^m$ is linear if
(i) $L(x + y) = L(x) + L(y)$
(ii) $L(\alpha x) = \alpha L(x)$
for all $x, y \in Rn$, $\alpha \in R$

Types of linear algebra:
Eigenvalues
Eigen values are associated with eigenvectors in Linear algebra. Both terms are used in the analysis of linear transformations. Eigenvalues are unique scalar values associated with a set of linear equations, commonly appearing in matrix equations. Eigenvectors, also known as characteristic roots, are non-zero vectors that, under a linear transformation, change only in magnitude by a scalar factor.

And the corresponding factor which scales the eigenvectors is called an eigenvalue.

Eigenvalue Definition
Eigenvalues are a distinct set of scalars linked to systems of linear equations, primarily in matrix equations. Derived from the German word "eigen," meaning "proper" or "characteristic," eigenvalues are also referred to as characteristic values, characteristic roots, proper values, or latent roots. Simply put, an eigenvalue is a scalar used to scale or transform an eigenvector. The fundamental equation governing this relationship is:$Ax = \lambda x$

The number or scalar value "λ" is an eigenvalue of A. In Mathematics, an eigenvector corresponds to the real non zero eigenvalues which point in the direction stretched by the transformation whereas eigenvalue is considered as a factor by which it is stretched.If an eigenvalue is negative, it indicates that the direction of the transformation is reversed.

For every real matrix, there is an eigenvalue. Sometimes it might be complex. The existence of eigenvalues for complex matrices is guaranteed by the fundamental theorem of algebra.

The key idea is to avoid all the complications presented by the matrix A. Suppose the solution vector u(t) stays in the direction of a fixed vector x. Then we only need to find the number (changing with time) that multiplies x. A number is easier than a vector. We want "eigenvectors" x that don't change direction when you multiply by A.

EigenVectors:
Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and x is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of x.

An Eigenspace of vector x consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "$n \times n$" matrix and λ is an eigenvalue of matrix A, then x, a non-zero vector, is called as eigenvector if it satisfies the given below expression;

$$Ax = \lambda x$$

x is an eigenvector of A corresponding to eigenvalue, λ.

Note:
- There could be infinitely many Eigenvectors, corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.

Eigenvalues of a Square Matrix

Suppose, $A_{n\times n}$ is a square matrix, then [A- λI] is called an Eigen or characteristic matrix, which is an indefinite or undefined scalar. Where determinant of Eigen matrix can be written as |A- λI| = 0 is the Eigen equation or characteristics equation, where "I" is the identity matrix. The roots of an Eigen matrix are called Eigen roots.

Eigenvalues of a triangular matrix and diagonal matrix are equivalent to the elements on the principal diagonals. But eigenvalues of the scalar matrix are the scalar only.

Properties of Eigenvalues

- Eigenvectors with Distinct Eigenvalues are Linearly Independent
- Singular Matrices have Zero Eigenvalues
- If A is a square matrix, then λ = 0 is not an eigenvalue of A
- For a scalar multiple of a matrix: If A is a square matrix and λ is an eigenvalue of A. Then, aλ is an eigenvalue of aA.
- For Matrix powers: If A is square matrix and λ is an eigenvalue of A and n≥0 is an integer, then λ^n is an eigenvalue of A^n.
- For polynomials of matrix: If A is a square matrix, λ is an eigenvalue of A and p(x) is a polynomial in variable x, then p(λ) is the eigenvalue of matrix p(A).
- Inverse Matrix: If A is a square matrix, λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}
- Transpose matrix: If A is a square matrix, λ is an eigenvalue of A, then λ is an eigenvalue of A^t

Eigenvalues of 2 x 2 Matrix

Let us have a look at the example given below to learn how to find the eigenvalues of a 2 x 2 matrix.

Find the eigenvalues of the 2 x 2 matrix

1} solve eigen values and eigen vectors of $= \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$

?

Solution:

Given,

$$A = \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$$

Using the characteristic equation,

Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be the 2 x 2 identity matrix.

|A – λI| = 0

$$\left|\begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} -\lambda & -2 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda(4 - \lambda) - (-2)(0) = 0$$

$$-4\lambda + \lambda^2 = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0 \text{ or } \lambda - 4 = 0$$

Thus, λ = 0 or λ = 4

Hence, the two eigenvalues of the given matrix are λ = 0 and λ = 4.

Go through the following problem to find the Eigenvalue of 3 x 3 matrix.

1} solve eigen values and eigen vectors of $= \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

?

Solution:

Given,

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Using the characteristic equation,

Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be the 2 x 2 identity matrix.

|A – λI| = 0

λ(4 – λ) – (-2)(0) = 0

-4λ + λ^2 = 0

λ(λ – 4) = 0

λ = 0 or λ – 4 = 0

Thus, λ = 0 or λ = 4

Hence, the two eigenvalues of the given matrix are λ = 0 and λ = 4.

Go through the following problem to find the Eigenvalue of 3 x 3 matrix.

Example 2:

Find all eigenvalues and corresponding eigenvectors for the matrix A if

$$\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: Given that

$$\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 2-\lambda & -3 & 0 \\ 2 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$(2-\lambda) \det\begin{bmatrix} 5-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} - (-3) \det \begin{bmatrix} 2 & 0 \\ 0 & 3-\lambda \end{bmatrix} + 0 . \det\begin{bmatrix} 2 & -5-\lambda \\ 0 & 0 \end{bmatrix} = 0$$

$$(2-\lambda)(\lambda^2+2\lambda-15)-(-3)\cdot 2(-\lambda+3)+0\cdot 0=0$$

$$-\lambda^3+13\lambda-12=0$$

$$-(\lambda-1)(\lambda-3)(\lambda+4)=0$$

Theeigenvaluesare:

$\lambda = 1, \lambda = 3, \lambda = -4$

Eigenvectorsfor$\lambda = 1$

$$\det \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$(A-1I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 3y = 0$

$z = 0$

X=3y

Z=0

If y=1

X=3,y=1,z=0

Therefore for λ =1: $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Therefore for λ =-3: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Therefore for λ =4: $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

CONCLUSION

In conclusion, eigenvalues and eigenvectors are important ideas in linear algebra having many uses in science, engineering, data analysis, and mathematics. Here are the main points to remember in relation to eigenvalues and eigenvectors:square matrices are related to eigenvalues and eigenvectors, according to the definition. An eigenvector of a matrix A is a nonzero vector v such that when A is multiplied by v, the outcome is a scaled version of v: = Av=v, where is the eigenvalue associated with v.Every eigenvalue has an accompanying eigenvector, and together they make up an eigenvalueeigenvector pair. Eigenvectors with different values for the same eigenvalue are scalar multiples of one another.Eigenvalues and eigenvectors are important in linear transformations because they provide light on how these transformations behave when they are represented by matrices. They illustrate how certain vector transformations rotate or scale.The method of diagonalization entails locating a diagonal matrix D and an invertible matrix P such that 1 = P 1 AP=D, where A is the original matrix. If and only if there is a full set of linearly independent eigenvectors, then diagonalization is conceivable. Exponentiation and powers of matrices are made simpler by this approach.The characteristic equation det ( ) = 0 det(AI)=0, where I is the identity matrix, may be used to determine a matrix's eigenvalues. The eigenvalues are the answers to this equation.In physics, eigenvalues and eigenvectors are often employed to address issues related to dynamic systems, quantum mechanics, and classical mechanics. Principal Component Analysis (PCA) is a data analysis method that use eigenvectors to minimize the dimensionality of data while maintaining the greatest amount of information. It is commonly used in image processing, machine learning, and statistics.Engineering and control theory employ eigenvalues to analyze stability to assess the stability of dynamic systems. If all of the eigenvalues have negative real components, the system is said to be stable. The technique of representing a matrix as a linear combination of its eigenvectors and eigenvalues is known as Eigen decomposition. Numerous applications, such as the solution of systems of linear differential equations, benefit from this decomposition.Eigenvectors of orthogonal matrices have the unique characteristic of forming an orthonormal basis. Numerous applications, including quantum physics and computer graphics, make use of this A Textbook of Advance Calculus, Vectors & Numerical Analysis 29 characteristic.In conclusion, eigenvalues and eigenvectors are important mathematical ideas that provide light on data analysis and linear transformations. They are crucial resources for comprehending and resolving challenging issues in the domains of science, engineering, and computer science due to their wide variety of applications.

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